Preferences and Performance in Simultaneous First-Price Auctions: A Structural Analysis

Matthew Gentry† Tatiana Komarova‡ Pasquale Schiraldi§

November 2018

Abstract

Motivated by the empirical prevalence of simultaneous bidding across a wide range of auction markets, we develop and estimate a structural model of strategic interaction in simultaneous first-price auctions when objects are heterogeneous and bidders have preferences over combinations. We establish non-parametric identification of primitives in this model under standard exclusion restrictions, providing a basis for both estimation and testing of preferences over combinations. We then apply our model to data on Michigan Department of Transportation (MDOT) highway procurement auctions, quantifying the magnitude of cost synergies and evaluating the performance of the simultaneous first-price mechanism in the MDOT marketplace.

∗We are grateful to Philip Haile, Ali Hortacsu, Ken Hendricks, Paul Klemperer, Paulo Somaini and Balazs Szentes for their comments, suggestions and insights. We also thank seminar and conference participants at the AEA Winter Meetings 2015, CEPR 2015, the NBER Winter Meetings 2016, the University of Wisconsin (Madison), the University of Zurich, the University of Leuven, Oxford University, City University, Cornell University, Université Paris 1, Yale University, Vanderbilt University, Duke University, Johns Hopkins University, Mannheim University, Emory University and the University of Maryland for helpful discussion. We thank Joachim Groeger and Paulo Somaini for graciously sharing their data. Financial support from ESRC Grant #ES/N000056/1 is gratefully acknowledged. An earlier draft of this paper was circulated under the title “Simultaneous First-Price Auctions with Preferences over Combinations: Identification, Estimation and Application.”

†Florida State University, mgentry@fsu.edu
‡London School of Economics, t.komarova@lse.ac.uk
§London School of Economics and CEPR, p.schiraldi@lse.ac.uk
1 Introduction

Simultaneous bidding in multiple first-price auctions is a commonly occurring but rarely discussed phenomenon in many real-world auction markets.\(^1\) In environments where values over combinations are non-additive in the set of objects won, bidders must account for the possibility of winning multiple auctions at the time of bidding. This in turn substantially alters the strategic bidding problem compared to the standard first price auction, with ambiguous welfare implications depending on the importance of synergies (either positive or negative) among objects. We develop a structural model of bidding in simultaneous first-price auctions and study identification and estimation in this framework. We then apply our methodology to estimate cost synergies arising in Michigan Department of Transportation (MDOT) highway procurement auctions, using the resulting estimates to analyze revenue and efficiency performance of the simultaneous first-price mechanism in this application.\(^2\)

To illustrate the policy questions arising in simultaneous multi-object auctions, note that given a set of \(L\) heterogeneous objects for sale, bidders’ preference structure could in principle be as complex as a complete \(2^L\)-dimensional set of signals describing the valuations \(i\) assigns to each of the \(2^L\) possible subsets of objects. Meanwhile, the simultaneous first-price mechanism allows bidders to submit (at most) \(L\) individual bids on the \(L\) objects being sold. Consequently, the simultaneous first-price auction format is necessarily inefficient—the “message space” (standalone bids) is insufficiently rich to allow bidders to express their true preferences. The auctioneer could alleviate this “message space” problem by, for instance, allowing combination bids, but these still need not produce efficient allocations, and may

---

\(^1\)To underscore the prevalence of simultaneous bidding in applications, note that many widely studied first-price marketplaces in fact exhibit simultaneous bids. Concrete examples include markets for highway procurement in many US states (Krasnokutskaya (2011), Somaini (2015), Li and Zheng (2009), Groeger (2014), many others), snow-clearing in Montreal (Flambard and Perrigne (2006)), recycling services in Japan (Kawai (2011)), cleaning services in Sweden (Lunander and Lundberg (2013)), oil and drilling rights in the US Outer Continental Shelf (Hendricks et al. (2003)), and to a lesser extent US Forest Service timber harvesting (Lu and Perrigne (2008), Li and Zhang (2010), Athey et al. (2011), many others).

\(^2\)This paper focuses on complementarities arising when auctions are run simultaneously. This complements the literature on potential linkages in valuations over time, e.g. Balat (2015), De Silva (2005), De Silva et al. (2005), Groeger (2014) and Jofre-Bonet and Pesendorfer (2003) among others.
involve substantial practical costs; see, e.g., Cramton et al. (2006) for a review. Hence in evaluating the simultaneous first-price format, it is first necessary to assess potential welfare and revenue effects of simultaneous bidding, about which little is presently known.

We develop a structural empirical model of bidding in simultaneous first-price auctions when objects are heterogeneous and bidders have non-additive preferences over combinations, to our knowledge the first in the literature. We represent the total value bidder $i$ assigns to each combination as the sum of two components: the sum of $i$’s standalone valuations for each object in the combination individually, plus a combination-specific complementarity (either positive or negative) capturing the incremental change in value $i$ associates with winning the combination as a whole. We interpret standalone valuations as private information drawn independently across bidders conditional on observables, and complementarities as deterministic functions of observables.\(^3\) We find this framework natural in a variety of procurement contexts—when, for instance, non-additivity in preferences can be represented as the expectation over a cost shock realized following a multiple win. Furthermore, and crucially, our framework collapses immediately to the standard separable model when complementarities are zero, supporting formal testing of this hypothesis.

Building on this framework, we make four main contributions to the literature on structural analysis of auction markets. First, we establish a new set of identification results applicable even when complementarities are non-zero. We start by showing that optimal behavior in this environment yields an inverse bidding system non-parametrically identified up to the unknown function describing complementarities, which in turn collapses to the standard inverse bidding function of Guerre et al. (2000) when complementarities are zero. Under natural exclusion restrictions—namely, that marginal distributions of standalone valuations are invariant either to characteristics of rival bidders or characteristics of other objects—we then translate this inverse bidding system into a system of linear equations in unknown bids.

\(^3\)Note that this structure does not restrict dependence between $i$’s standalone valuations for different objects in the market. We view this flexibility as critical, as in practice we expect $i$’s standalone valuations to be positively correlated.
der complementarities, with excludable variation in competition and other characteristics yielding non-parametric identification of these.

Second, we develop a three-step procedure by which to estimate primitives in our structural model. First, in Step 1, we estimate the multivariate joint distribution of bids as a function of bidder- and auction-level characteristics. Next, in Step 2, we exploit the invariance of first moments of standalone valuations with respect to the set of excluded instruments discussed above to generate a set of moment conditions based on which we estimate cross-project complementarities. Finally, in Step 3, we map these estimated complementarities through the inverse bidding system derived in Step 1 to estimate standalone project completion costs for each bidder.

Third, we apply our structural framework to analyze simultaneous bidding in Michigan Department of Transportation (MDOT) highway procurement markets. We view this market as prototypical of our target application: large numbers of projects are auctioned simultaneously (an average of 45 per letting round in our 2005-2015 sample period), more than half of bidders bid on at least two projects simultaneously (with an average of 2.7 bids per round across all bidders in the sample), and combination and contingent bidding are explicitly forbidden. Within this marketplace, we show that factors such as size of other projects, number of bidders in other auctions, and the relative distance between projects have substantial reduced-form impacts on i’s bid in auction l, a finding hard to rationalize in standard separable models. We then apply the estimation algorithm described above to recover structural estimates of primitives. We find substantial complementarities in this application: a combination win may generate anything from approximately 13 percent cost savings to approximately 6.6 percent cost increases depending on project characteristics, with large, heterogeneous, and overlapping projects more likely to be substitutes.

Finally, building on our structural estimates, we measure potential inefficiencies associated with the simultaneous first price auction design. Toward this end, we compare the simultaneous first-price auction used in the MDOT marketplace with a simple efficient com-
binatorial benchmark: the Vickrey-Clarke-Groves (VCG) mechanism. As expected, the VCG mechanism yields lower social costs: our estimates suggest total social gains of approximately 4.5 percent, with relatively larger gains in lettings with larger complementarities. Interestingly, however, the counterfactual VCG auction also increases MDOT’s procurement costs by about 13.7 percent. In other words, even in the presence of substantial complementarities, the efficiency gains from the VCG mechanism are relatively small, while the costs to MDOT are relatively large. This is, to our knowledge, the first formal comparison of the simultaneous first-price and VCG mechanisms, and may help to explain the popularity of the simultaneous first-price format in practice.

While this is to our knowledge the first structural analysis of bidding in simultaneous first-price auctions, our work builds on a small but growing structure literature analyzing combinatorial auctions. Cantillon and Pesendorfer (2006) analyze combinatorial first-price sealed-bid auctions for London bus routes, using the possibility of package bidding to identify bidder preferences over combinations. More recently, Kim et al. (2014) have extended this methodology to analyze the large-scale combinatorial auctions used in procurement of Chilean school meals. A key source of identification in these combinatorial settings is observation of package bids, which are directly informative regarding relative preferences for specific combinations. Since, by construction, we observe only standalone bids, identification in our simultaneous first-price setting is a substantially different (and more challenging) problem, for which we develop a novel solution. Finally, in the context of FCC simultaneous

---

4 We also explored other leading combinatorial mechanisms, such as the descending proxy auction of Ausubel and Milgrom (2002). By constructing, these lead to the same efficient allocation as the VCG auction, and in preliminary tests they also led to very similar expected revenue. For this reason, we chose to focus for simplicity on the VCG auction.

5 More generally, this analysis contributes to the growing literature that aims to understand the performance of different auction formats in procurement auctions. Other studies in this area includes among others: Athey et al. (2011) who compare open versus sealed bid auctions for timber harvesting contracts, Lewis and Bajari (2011) who compare first price versus scoring rules with time incentives for the procurement of roadwork contracts, and Decarolis (2017) who studies first price auctions with ex-post screening and average-bid auctions in the Italian procurement.

6 Although only tangentially related to our problem, there is also a growing empirical literature on multi-unit auctions, which focus on markets for homogeneous, divisible goods like electricity and treasury bills. See e.g. Fevrier et al. (2004); Chapmat et al. (2007); Kastl (2011); Hortacsu and Puller (2008); Hortacsu and McAdams (2010) and Hortacsu (2011); Wolak (2007); and Reguant (2014).
ascending auctions, Bajari and Fox (2013) estimate the deterministic component of bidder valuations. Specifically, they exploit the assumption that the allocation of licenses is pairwise stable in matches, a condition which need not hold in the simultaneous first-price setting we consider here.

Paralleling these structural studies, there is also a small reduced-form literature seeking to quantify the role of preferences over combinations in multi-object auctions. Ausubel et al. (1997) and Moreton and Spiller (1998) measure synergy effects in FCC spectrum auctions. Lunander and Lundberg (2013) empirically compare combinatorial and simultaneous first-price auctions in a Swedish market for internal cleaning services, finding that bidders inflate their standalone bids in combinatorial auctions relative to first-price auctions but that this does not significantly affect the procurer’s final costs. De Silva (2005) and De Silva et al. (2005) analyze spatial synergies in Oklahoma Department of Transportation highway procurement auctions, finding that previous winners participate more often and bid more aggressively in subsequent nearby projects. These findings are consistent with the hypothesis of spatial synergies in procurement, motivating the structural model we consider here.

Finally, from a more theoretical perspective, there have been several studies analyzing strategic interaction in stylized models involving simultaneous first-price auctions; see, for example, Szentes and Rosenthal (2003) and Ghosh (2012) among others. Gentry et al. (2017) study existence and proprieties of equilibrium in a setting closely paralleling that studied here. There is also a substantial literature analyzing properties of various combinatorial auction mechanisms: Ausubel and Milgrom (2002), Ausubel and Cramton (2004), Cramton (1997, 2006), Krishna and Rosenthal (1996), Klemperer (2008, 2010), Milgrom (2000b, 2000a), and Rosenthal and Wang (1996), to mention just a few. Detailed surveys of this literature are given in De Vreis and Vorha (2003) and Cramton et al. (2006).

7There is also a growing theoretical literature on simultaneous first-price auctions in computer science; see Feldman et al. (2012) and Syrgkanis (2012) among others. This literature focuses primarily on deriving bounds on the “Bayesian price of anarchy,” or fractional efficiency loss, in simultaneous first-price auction markets. Results in this literature are largely restricted to settings with negative complementarities, and even in these settings bounds tend to be wide (e.g. Feldman et al. (2012) show that Bayesian Nash equilibrium captures at least half of total social surplus).
The rest of this paper is organized as follows. Section 2 outlines the model of simultaneous first-price auctions on which our structural analysis is built. Section 3 studies identification in this model. Section 4 describes the Michigan Department of Transportation (MDOT) highway procurement marketplace, and Section 5 presents our structural results. Section 6 compares MDOT’s simultaneous first-price format with a combinatorial VCG mechanism. Finally, Section 7 concludes. Additional results are collected in a set of technical appendices: Appendix A collects technical proofs, Appendix B extends our framework to incorporate entry, and Appendices C-F present extended identification and testing results.

2 Empirical framework

Consider a population of simultaneous first-price lettings. In each letting $t$, a set $N_t = \{1, ..., N_t\}$ of risk-neutral bidders compete for (subsets of) a set $L_t = \{1, ..., L_t\}$ of objects allocated via separate but simultaneous first-price auctions. Each bidder $i \in N_t$ participates in a set of auctions, $L_{it} \subset L_t$, submitting a scalar bid $b_{itl}$ in each auction $l$ in which she participates. Bidding is simultaneous and objects are awarded auction by auction: the high bidder in auction $l$ wins object $l$ and pays her bid, with ties broken independently across bidders and auctions. Let $L_{it}$ denote the number of auctions in which bidder $i$ is participating, and $b_{it} \equiv (b_{itl})_{l \in L_{it}}$ denote the $L_{it} \times 1$ vector of bids submitted by $i$ in letting $t$.

For each letting $t$, the econometrician observes the following data. First, for each object $l = 1, ..., L_t$ auctioned in letting $t$, the econometrician observes a vector of characteristics $X_{lt}$ describing this object. Second, for each bidder $i$ active in letting $t$, the econometrician observes bidder $i$’s bid vector $b_{it}$, participation set $L_{it}$, and a vector of generic bidder characteristics $Z_{it}$. In what follows, let $X_t \equiv (X_{1,t}, ..., X_{L_t,t})$ describe characteristics of all objects auctioned in letting $t$, and $Z_t \equiv (Z_{1t}, ..., Z_{N_t,t})$ describe characteristics of all active bidders.

Following Cantillon and Pesendorfer (2006) and Bajari and Fox (2013), we analyze bidding in the simultaneous first-price auction taking participation as given. That is, we take
the endogenous outcome of interest to be the bid vectors \((b_{1t})_{i=1}^{N_t}\) submitted by each bidder, conditional on auction characteristics \(X_t\), bidder characteristics \(Z_t\), and participation sets \((\mathcal{L}_{it})_{i=1}^{N_t}\). We view this as a natural, and arguably necessary, first step toward understanding simultaneous first-price auction markets: here, as elsewhere, one cannot analyze participation without understanding bidding. Importantly, however, one can also view our analysis as applying to bidding within several canonical two-stage entry and bidding games. We return to this point in more detail below, after describing the structure of bidder preferences.

For concreteness, we follow many prior studies on highway procurement auctions, e.g. Bajari and Ye (2003), Krasnokutskaya (2011), and Krasnokutskaya and Seim (2011) among many others, in assuming that bidders observe the participation structure \((\mathcal{L}_{it})_{i=1}^{N_t}\) at the time of bidding. We note, however, that our identification analysis applies equally when bidders observe only the set of potential participants in each auction; e.g., the set of planholders as in Li and Zheng (2009). In this case, one would simply reinterpret \(\mathcal{L}_{it}\) as the set of auctions in which \(i\) is a potential participant, then proceed as we describe below.

In either case, to streamline notation, we adopt the convention that bidder \(i\)’s characteristics \(Z_{it}\) include her participation set \(\mathcal{L}_{it}\). From the perspective of both bidders and the econometrician, the common-knowledge observables \((X_t, Z_t)\) this fully characterize letting \(t\). Since all subsequent discussion will apply conditional on specific realizations of \((X_t, Z_t)\), in what follows we suppress the subscript \(t\).

Our baseline model turns on two sets of structural assumptions: the first regarding bidder preferences, the second regarding equilibrium behavior. In this section, we describe each of these in turn. We discuss extensions of this baseline model to settings with endogenous participation, unobserved heterogeneity, and richer preferences in Section 3.6.

2.1 Bidder preferences

By construction, if bidder \(i = 1, \ldots, N\) participates in \(L_i \geq 1\) auctions, then she may win any of \(2^{L_i}\) possible combinations of objects. In what follows, we index these possible combinations...
with an $L_i \times 1$ outcome vector $\omega_i$, where $\omega_{il} = 1$ if object $L_{il}$ is allocated to bidder $i$ and $\omega_{il} = 0$ otherwise. We represent the set of all $2^{L_i}$ combinations possible for bidder $i$ with a $2^{L_i} \times L_i$ outcome matrix $\Omega_i$, where each row of $\Omega_i$ corresponds to a distinct outcome $\omega_i$. For example, if $L_i = 2$, then $\Omega_i$ would be given by

$$\Omega_i = \begin{bmatrix}
0 & 0 \\
1 & 1 \\
0 & 1 \\
0 & 1
\end{bmatrix}.$$ 

Equivalently, one may view each outcome $\omega_i$ as the binary representation of some integer in the set $\{1, \ldots, 2^{L_i}\}$ indexing $i$’s possible combinations, with $\Omega_i$ collecting all such binary representations. With slight abuse of notation, we therefore use the shorthand “$\omega \in \Omega_i$” to indicate that outcome $\omega$ is possible for bidder $i$.

Bidders have preferences over combinations of objects, which need not be additive over elements in the combination. Specifically, to each potential outcome $\omega \in \Omega_i$, bidder $i$ associates a private combinatorial valuation $Y_{i, \omega}$, which she receives in the event that this outcome is realized. Let $Y_i \equiv [Y_{i, \omega}]_{\omega \in \Omega_i}$, a $2^{L_i} \times 1$ vector, collect $i$’s combinatorial valuations $Y_{i, \omega}$ for all possible outcomes $\omega \in \Omega_i$. For simplicity, and without loss of generality, we normalize the value of winning nothing to zero: $Y_i^0 = 0$.

Let bidder $i$’s *standalone valuation* for object $l$, denoted $V_{il}$, be the valuation $i$ assigns to the outcome “$i$ wins object $l$ alone”. Accordingly, let $i$’s *standalone valuation vector*, denoted $V_i$, be the $L_i \times 1$ vector describing $i$’s standalone valuations for each object in her participation set: $V_i \equiv [V_{il}]_{l=1}^{L_i}$. Finally, let $K_{i, \omega}$ denote the *complementarity* $i$ between objects in combination $\omega \in \Omega_i$, defined as the difference between $i$’s combinatorial valuation $Y_{i, \omega}$ and the sum of $i$’s standalone valuations for objects won under $\omega$:

$$K_{i, \omega} = Y_{i, \omega} - \omega^T V_i.$$

9
Let $K_i \equiv [K_i^\omega]_{\omega \in \Omega_i}$ be the $2^{L_i} \times 1$ vector containing the complementarities associated by $i$ with each possible outcome $\omega \in \Omega_i$. Note that, by construction, we have

$$Y_i \equiv \Omega_i V_i + K_i.$$ 

We may thus equivalently represent bidder $i$'s preferences in terms of the pair $(V_i, K_i)$, where $V_i$ describes $i$'s valuations for each object individually, while $K_i$ reflects departures from additivity in $i$'s preferences over combinations. In particular, our model reduces to the canonical additively separable case if and only if $K_i = 0$ for all $i$.

As usual, we interpret standalone valuation vectors $V_i$ as stochastic and private information for each bidder $i$. We further assume that standalone valuation vectors $(V_1, ..., V_N)$ are distributed independently across bidders conditional on observables:

**Assumption 1** (Independent private standalone valuations). For each bidder $i = 1, ..., N$, standalone valuations $V_i$ are distributed according to a joint c.d.f. $F_i(\cdot | Z, X)$, with $V_i \perp V_j$ for all $j \neq i$, and $F_i(\cdot | Z, X)$ common knowledge.

While standalone valuations are stochastic private information, we model complementarities $K_i$ as determined by observables. We view this structure as natural in applications such as highway contracting, snow cleaning (Flambard and Perrigne (2006)), recycling (Kawai (2011)), and cleaning (Lunander and Lundberg (2013)), where factors such as capacity constraints, distance between projects, timing of projects, or types of work are the main considerations motivating analysis of complementarities.

**Assumption 2** (Deterministic complementarities). For all $i = 1, ..., N$ and each combination $\omega \in \Omega_i$, $K_i^\omega = \kappa_i^\omega(Z, W_i^\omega)$, where $W_i^\omega$ is an observed vector of characteristics of the combination $\omega \in \Omega_i$, and $\kappa_i^\omega(Z, W_i^\omega)$ is common knowledge to bidders.

In what follows, let $W_i \equiv [W_i^\omega]_{\omega \in \Omega_i}$ collect characteristics of all potential combinations.
for bidder \(i\), and \(W \equiv (W_1, ..., W_N)\) collect characteristics of all combinations played by all bidders. With slight abuse of notation, we write \(K_i = \kappa_i(Z, W_i)\) to indicate the vectorization of Assumption 2 across all potential combinations relevant for bidder \(i\).

We denote combination characteristics \(W\) and auction characteristics \(X\) distinctly for conceptual clarity, although in practice \(W\) will typically be derived through transformation of \(X\). For example, in our highway procurement application, \(X\) includes factors such as project size, project location, and type of work in each project, whereas \(W\) includes factors such as distance between projects, sum of project sizes, and the degree of schedule overlap among projects in each combination. We view these as close proxies for the main factors inducing potential nonseparability in this market, motivating Assumption 2 as noted above.

Taken together, Assumptions 1 and 2 embed the canonical separable IPV model within a substantially richer framework allowing both flexible nonparametric complementarities and arbitrary dependence between the elements of \(V_i\) for each bidder \(i\). We view the latter as an essential empirical complement to the former, since “reduced form” correlation in \(i\)'s bids could be driven either by complementarities in \(i\)'s preferences or by dependence in \(i\)'s valuations. By leaving such dependence unrestricted, we focus cleanly on identification of nonadditivities per se, even when bidder \(i\)'s valuations are correlated across auctions.

### 2.2 Equilibrium behavior

Let \(M \equiv (Z, X, W)\) denote the set of market characteristics observed to both bidders and the econometrician. Let \(V_i \subset \mathbb{R}_+^{L_i}\) denote the support of \(V_i\) for bidder \(i = 1, ..., N\), and let \(B_l \subset \mathbb{R}_+\) denotes the set of feasible bids in auction \(l = 1, ..., L\). A pure strategy for bidder \(i\) in market \(M\) is then a mapping \(\sigma_i^M : V_i \to B_i\), where \(B_i \equiv \times_{l \in L_i} B_l\) denotes \(i\)'s action space in the simultaneous bidding game.\(^9\) Let \(\sigma^M = (\sigma_1^M, ..., \sigma_N^M)\) denote a strategy profile for all bidders in market \(M\), and \(\sigma_{-i}^M\) denote a strategy profile for all rivals of bidder \(i\).

Building on the first-order approach of Guerre et al. (2000), we base identification on

\(^9\)We focus on pure strategies for expositional simplicity, but this is without essential loss of generality; all results below apply equally when bidders play mixed strategies.
necessary conditions for best-response behavior in simultaneous first-price auctions. For this analysis to proceed, we require the following assumptions on bidder behavior:

**Assumption 3.** For each market structure $M$, the distribution of bids observed at $M$ are generated by a strategy profile $\sigma^M$ which is a Bayesian Nash equilibrium of the simultaneous bidding game. Furthermore, for each $M$, only one strategy profile $\sigma^M$ is played.

When complementarities are zero, existence of a pure strategy equilibrium is immediate and uniqueness follows under regularity conditions (Lebrun (1999)). More generally, with nonzero complementarities, existence of a pure strategy equilibrium in any discrete bid space follows from results in Milgrom and Weber (1985). Analysis of equilibrium with arbitrary complementarities in continuous bid spaces would be a fundamental breakthrough in its own right, and as such is well beyond the scope of this paper.\textsuperscript{10} In this respect, we parallel many studies on complex auction games, in which either existence (Bajari and Fox (2013) on spectrum auctions, Ausubel and Milgrom (2002) on proxy auctions) or uniqueness (Jofre-Bonet and Pesendorfer (2003), Roberts and Sweeting (2013), Somaini (2015) and references therein) is assumed as it cannot be guaranteed in general.\textsuperscript{11}

To leverage necessary conditions for optimal behavior, we require only the hypotheses on equilibrium behavior stated in Assumption 3. For such an analysis to yield point (rather than partial) identification of model primitives, we further require equilibrium behavior to satisfy the following additional regularity conditions:

\textsuperscript{10}“Fundamental” in the sense that existing theoretical tools appear inadequate to study existence in settings with complementarities. As in multi-unit auctions, the presence of both multidimensional bids and multidimensional types leads to failure of classical differential-equations approaches to Bayes-Nash equilibrium. Monotonicity-based methods widely used in multi-unit auctions—e.g. Athey (2004), McAdams (2006), and Reny (2011)—can be applied in special cases, but do not apply at the level of generality we consider here. Other approaches—e.g. that of Jackson et al. (2002) applied in Cantillon and Pesendorfer (2006)—deliver generalizations of Bayes-Nash equilibrium, but not Bayes-Nash equilibrium itself. See Gentry et al. (2017) for a detailed discussion of these issues, plus results on equilibrium existence in some special cases.

\textsuperscript{11}We note, however, that almost every real-world bid space is ultimately discrete: for instance, if bidders must bid in pennies, then existence is guaranteed as noted above. In this sense, we see existence as of more theoretical than practical concern. In the main text, we follow the literature’s overwhelming convention of interpreting bid spaces as “approximately continuous,” and proceed to analyze identification. Appendix E provides a more general partial identification analysis applicable in settings where discreteness is viewed as empirically important.
**Assumption 4.** For each observed market structure $M$, the equilibrium strategy profile $\sigma^M$ is such that (i) the joint cumulative distribution function of bids is absolutely continuous, and (ii) for any auction $l = 1, ..., L$ and any bidders $i, j$ active in auction $l$, the marginal distributions of bids $b_{il}, b_{jl}$ have common support.

As above, under the null of separability ($K_i = 0$), these properties follow immediately from standard regularity conditions; when $K_i \neq 0$, we require them as assumptions. In practice, the main role of Assumption 4 is to ensure that marginal bid distributions are atomless, which in turn permits extension of the Guerre, Perrigne and Vuong (2000) first-order approach to settings with simultaneous auctions. In Appendix E, we extend our identification analysis to accommodate violations of Assumption 4. The main ideas of this extension closely parallel those developed in the main text, although only yielding partial, not point, identification of model primitives.

## 3 Nonparametric identification

We analyze identification based on a large number of simultaneous first-price auction markets. By hypothesis, for each market, the econometrician observes the vector of common-knowledge covariates $M$ characterizing bidders, objects, and combinations in the market, together with the bid vectors $(b_i)_{i=1}^N$ submitted by each bidder active in the market. The identification problem is to recover the nonparametric primitives $F_i(\cdot|Z,X)$ and $\kappa_i(Z,W_i)$ for each bidder $i$ active in market $M$.

In analyzing this problem, we adopt the following notation. For each bidder $i = 1, ..., N$, let $G_i(\cdot|M)$ be the joint cumulative distribution function of the $L_i \times 1$ bid vector $b_i$ submitted by $i$ conditional on market characteristics $M$, and $g_i(\cdot|M)$ be the corresponding conditional joint density. Taking equilibrium rival strategies $\sigma^M_i$ as given, suppose that bidder $i$ submits bid $b_i \in B_i$. For each auction $l \in L_i$, let $\Gamma_i(b_i|M)$ denote this bidder’s marginal probability of winning auction $l$, and for each combination $\omega \in \Omega_i$, let $P_i^\omega(b_i|M)$ denote the joint
probability that she wins combination \( \omega \), both interpreted as functions of \( i \)’s bid vector \( b_i \) taking rival strategies \( \sigma^M \) as given. Finally, let \( \Gamma_i(b_i|M) \equiv (\Gamma_i(b_i|M))_{l \in L_i} \), an \( L_i \times 1 \) vector, collect marginal win probabilities \( \Gamma_i(b_i|M) \) across auctions \( l \in L_i \), and let \( P_i(b_i|M) \equiv [P_i^\omega(b_i|M)]_{\omega \in \Omega_i} \), a \( 2^{L_i} \times 1 \) vector, collect combinatorial win probabilities \( P_i^\omega(b_i|M) \) across combinations \( \omega \in \Omega_i \). Note that, if there are no ties, then \( i \)’s marginal probability of winning auction \( l \), i.e. \( \Gamma_i(b_i|M) \), is simply the c.d.f. of the maximum rival bid in auction \( l \), evaluated at \( i \)’s bid \( b_i \). Furthermore, by construction, marginal win probabilities \( \Gamma_i(b_i|M) \) are related to combinatorial win probabilities \( P_i(b_i|M) \) by the identity \( \Gamma_i(b_i|M) \equiv \Omega_i^T P_i(b_i|M) \).

Observe that, under Assumption 3, \( G_i(\cdot|M) \) is identified directly for each \( i = 1, ..., N \), with identification of \( (G_i(\cdot|M))_{i=1}^N \) implying identification of \( P_i(\cdot|M) \) and \( \Gamma_i(\cdot|M) \) for all \( i \).

In what follows, we thus take bid distributions \( G_i(\cdot|M) \), marginal win probabilities \( \Gamma_i(\cdot|M) \), and combinatorial win probabilities \( P_i(\cdot|M) \) as known. We aim to recover \( F_i(\cdot|Z,X) \) and \( \kappa_i(Z,W) \) given knowledge of \( G_i(\cdot|M) \), \( P_i(\cdot|M) \), and \( \Gamma_i(\cdot|M) \) for all \( i = 1, ..., N \).

3.1 Identification intuition: a two-auction illustration

We begin with a simple example illustrating the main ideas of the identification argument. Consider a setting \( A \) in which a single “global bidder”, denoted \( i \), bids in two auctions against many symmetric “local bidders” who bid in single auctions only. Let \( n_{1A} \) and \( n_{2A} \) denote the number of local bidders in auctions 1 and 2 respectively. Finally, let \( v_i = (v_{i1}, v_{i2}) \) be \( i \)’s standalone valuations for each object individually, and \( \kappa_{12} \) be the complementarity that bidder \( i \) associates with winning both objects together.

For ease of illustration, in this example only, we further assume that we observe multiple lettings involving \textit{exactly the same} realization \( v_i \) of \( i \)’s standalone valuations \( V_i \). We thus aim to identify the scalar value realizations \( v_{i1} \) and \( v_{i2} \) (rather than the joint distribution \( F_i \) of \( V_i \)), together with the scalar complementarity \( \kappa_{12} \). In reality, of course, one observes (at best) only auctions involving the same latent \textit{distribution} of \( V_i \). The full identification argument therefore involves an additional integral over \( V_i \), which we make precise below.
The intuition of the argument, however, closely parallels that described here.

Suppose that all local bidders in each auction \( l = 1, 2 \) bid according to the same strategy. Let \( G_l(\cdot|A) \) denote the distribution of equilibrium bids submitted by each local bidder in auction \( l = 1, 2 \) in setting \( A \), with \( g_l(\cdot|A) \) the associated bid density. Taking \( v_i = (v_{i1}, v_{i2}) \) as given, bidder \( i \) chooses bids in each auction to solve

\[
\max_{b_1, b_2} \{(v_{i1} - b_1) \cdot G_1(b_1|A)^{n1A} + (v_{i2} - b_2) \cdot G_2(b_2|A)^{n2A} + \kappa_{12} \cdot G_1(b_1|A)^{n1A}G_2(b_2|A)^{n2A}\}.
\]

Taking first order conditions of this problem with respect to \( b_1 \) and \( b_2 \) and simplifying, we obtain the following system of first-order equations:

\[
v_{i1} = b_{1A} + \frac{G_1(b_{1A}|A)}{n_{1A}g_1(b_{1A}|A)} - \kappa_{12} \cdot G_2(b_{2A}|A)^{n2A},
\]

\[
v_{i2} = b_{2A} + \frac{G_2(b_{2A}|A)}{n_{2A}g_2(b_{2A}|A)} - \kappa_{12} \cdot G_1(b_{1A}|A)^{n1A}.
\]

Observe that all objects except \( \kappa_{12} \) on the right-hand side are directly identified. In other words, if we knew \( \kappa_{12} \), then we could recover \( (v_{i1}, v_{i2}) \) immediately. But since in practice we do not know \( \kappa_{12} \), the first order conditions (1)-(2) effectively represent a system of two linear equations in the three unknowns \( (v_{i1}, v_{i2}, \kappa_{12}) \). This system clearly will have multiple solutions, which implies that further restrictions are necessary for identification.

What additional information might identify \( \kappa_{12} \)? We consider three sources of potentially informative variation: in the set of competitors faced, in the characteristics of other objects, and in the characteristics of combinations. So long as (the marginal distribution of) at least one standalone valuation is held constant, variation in the first two factors can identify \( \kappa_{12} \), while variation in the third can identify changes in \( \kappa_{12} \). We next illustrate the additional identifying restrictions induced by variation in each of these factors in turn.

First consider variation in the set of competitors faced by \( i \). Specifically, consider a new setting, \( B \), identical to setting \( A \) except that the number of local bidders in auction 2 is
$n_{2B} > n_{2A}$, whereas $n_{1B} = n_{1A}$. The first-order condition (1) becomes

$$v_{i1} = b_{1B} + \frac{G_1(b_{1B}|B)}{n_{1B}g_{1}(b_{1B}|B)} - \kappa_{12} \cdot G_2(b_{2B}|B)^{n_{2B}},$$

Combining equations (1) and (3), we obtain the following expression for $\kappa_{12}$:

$$\kappa_{12} = \frac{b_{1B} - b_{1A} + \frac{G_1(b_{1B}|B)}{n_{1B}g_{1}(b_{1B}|B)} - \frac{G_1(b_{1A}|A)}{n_{1A}g_{1}(b_{1A}|A)}}{G_2(b_{2B}|B)^{n_{2B}} - G_2(b_{2A}|A)^{n_{2A}}}.$$ (4)

So long as $G_2(b_{2B}|B)^{n_{2B}} \neq G_2(b_{2A}|A)^{n_{2A}}$, which will generically be true since $i$ now faces more competition in auction 2, it follows from (4) that $\kappa_{12}$ is identified.

Second, consider variation in the characteristics of object 2. For simplicity, we model this as a shift to a environment $C$ where bidder $i$’s standalone valuation for auction 2 changes to $v'_{i2} \neq v_{i2}$, holding constant both $v_{i1}$ and $\kappa_{12}$. This could be achieved by, for instance, changing the physical distance between object 2 and bidder $i$, holding the distance between objects 1 and 2 constant. In this new environment $C$, the distribution of bids submitted by $i$’s local rivals will generally change, even though the competition structure is constant. Combining first order conditions, we again obtain an expression for $\kappa_{12}$ in terms of observables, closely paralleling equation (4) above. So long as the denominator $G_2(b_{2C}|C)^{n_{2C}} - G_2(b_{2A}|A)^{n_{2A}}$ in this expression in nonzero, $\kappa_{12}$ is identified.\footnote{Note that, in contrast to the case of variation in competition above, the rank condition $G_2(b_{2C}|C)^{n_{2C}} \neq G_2(b_{2A}|A)^{n_{2A}}$ is nontrivial: for instance, as we show in Appendix C, any characteristic which shifts all bidders’ standalone valuations for object 2 additively will also shift all bids in auction 2 additively, leaving relative winning probabilities unchanged. Nevertheless, essentially any shift in the auction environment except an additive shift in all valuations will induce $G_2(b_{2C}|C)^{n_{2C}} \neq G_2(b_{2A}|A)^{n_{2A}}$.}

Finally, consider variation in the characteristics of combinations, modeled as a shift to an environment $D$ where the competition structure and standalone valuations are as in environment $A$, but the complementarity bidder $i$ associates with winning both objects shifts to $\kappa'_{12} \neq \kappa_{12}$. Combining bidder $i$’s first order conditions for auction 1 in environments $A
and \(D\), we obtain the following expression for \(\kappa_{12}'\) in terms of identified objects and \(\kappa_{12}\):

\[
\kappa_{12}' = \frac{1}{G_2(b_{2D}|D)} \left( b_{1D} - b_{1A} + \frac{G_1(b_{1D}|D)}{n_1A_1g_1(b_{1D}|D)} - \kappa_{12} \cdot G_2(b_{2A}|A) \right).
\]

(5)

Suppose that the initial level \(\kappa_{12}\) is identified based on variation in either competition or characteristics of other auctions as above. Then equation (5) implies identification of \(\kappa_{12}'\) in the new environment \(D\). Furthermore, so long as bidder \(i\) wins auction 2 with positive probability, this holds without any further rank assumption. Variation in combination characteristics thus identifies, at minimum, changes in \(\kappa_i(\cdot)\) as a function of \(W\).\(^{13}\)

3.2 Nonparametric identification of standalone valuations up to complementarities

We now return to the general nonparametric identification problem: recovery of \((F_i, \kappa_i)_{i=1}^N\) given knowledge of \((G_i)_{i=1}^N\). Building on the simple example above, we begin by showing that bidder \(i\)'s unknown primitives \((F_i, \kappa_i)\) are nonparametrically identified up to \(\kappa_i\). We then explore identification of \(\kappa_i\) based on excludable variation in the set of competitors faced, the characteristics of other objects, and characteristics affecting complementarities.

Toward this end, consider the bidding problem faced by bidder \(i = 1, \ldots, N\) with preferences \((V_i, K_i)\) in market \(M\), where standalone valuations \(V_i\) are drawn privately from \(F_i(\cdot|X, Z)\) and complementarities \(K_i = \kappa_i(Z, W_i)\) are common knowledge as described above. By hypothesis, taking rival strategies \(\sigma_{-i}^M\) as given, this bidder optimally submits the \(L_i \times 1\) bid vector \(b_i \in B_i\) which maximizes her expected interim profit function

\[
\pi_i^M(b_i; V_i, K_i) = \Gamma_i(b_i|M)^T (V_i - b_i) + P_i(b_i|M)^T K_i,
\]

\(^{13}\)If \(v_{12}\) is also invariant to the factors affecting \(\kappa_{12}\), then it may be possible to say more: combining equation (5) with a similar equation based on first-order conditions in auction 2, we will obtain a system of two equations in the two unknowns \((\kappa_{12}, \kappa_{12}')\), which under suitable rank conditions will identify both \(\kappa_{12}\) and \(\kappa_{12}'\). But as this argument does not generalize easily, we do not pursue it further here.
where $\Gamma_i(b_i|M)^T(V_i - b_i)$ reflects the expected sum of bidder $i$’s canonical standalone payoffs over each auction individually, and $P_i(b_i|M)^T K_i$ reflects the change in $i$’s expected payoffs induced by non-additivities in her preferences over combinations.

Under Assumption 4, one can show that the interim function profit function (6) is differentiable in $b_i$ almost surely with respect to the measure on $B_i$ induced by $G_i(\cdot|M)$; we establish this formally in the proof of Proposition 1 below. Hence, under the hypothesis of equilibrium play, almost every bid $b_i \in B_i$ submitted by $i$ must satisfy necessary first-order conditions for an interior optimum:

$$\nabla_b \Gamma_i(b_i|M)^T(V_i - b_i) = \Gamma_i(b_i|M) - \nabla_b P_i(b_i|M)^T K_i.$$ (7)

Clearly, the system (7) is not invertible for $(V_i, K_i)$ jointly; we have only $L_i$ equations for $2^{L_i} - 1$ unknowns. But under our assumptions, this system is invertible for $V_i$ given $K_i$. In other words, under the hypothesis $K_i = \kappa_i(Z, W_i)$, there exists a unique candidate for $V_i$ at which $b_i$ satisfies first order necessary conditions for a best response:14

**Proposition 1.** Suppose that Assumptions 1-4 hold. Let $K_i$ be any candidate for bidder $i$’s unknown complementarity vector $\kappa_i(Z, W)$: i.e. any vector in $\mathbb{R}^{2L_i}$ whose first $L_i + 1$ components are zero.15 Then for almost every $b_i$ drawn from $G_i(\cdot|M)$, there exists a unique, identified vector $\xi_i(b_i|M; K_i)$ solving (7) under the hypothesis $\kappa_i(Z, W) = K_i$:

$$\xi_i(b_i|M; K_i) \equiv \Upsilon_i(b_i|M) - \Psi_i(b_i|M) \cdot K_i,$$ (8)

where $\Upsilon_i(b_i|M)$ is an identified $L_i \times 1$ vector defined by

$$\Upsilon_i(b_i|M) \equiv b_i + \nabla_b \Gamma_i(b_i|M)^{-1} \Gamma_i(b_i|M),$$ (9)

14Obviously, imposing sufficient conditions for $b_i$ to be a best response can only improve identification.

15These zero components correspond to the outcomes in which bidder $i$ wins either no objects ($\omega = (0, \ldots, 0)$) or one object ($\omega' \omega = 1$), for which complementarities are zero by construction.
and $\Psi_i(b_i|M)$ is an identified $L_i \times 2^{L_i}$ matrix defined by

$$\Psi_i(b_i|M) \equiv \nabla_b \Gamma_i(b_i|M)^{-1} \nabla_b P_i(b_i|M)^T.$$  \hspace{1cm} (10)

Furthermore, if $K_i = \kappa_i(Z,W)$, then $V_i = \xi_i(b_i|M; K_i)$ almost surely.

Proof. See Appendix A. \hfill \Box

Note that, interpreted as a function of $K_i$, $\xi_i(b_i|M; K_i)$ is affine in $K_i$ for all $b_i$ and $M$. The additive term $\Upsilon_i(b_i|M)$ in this affine function is the canonical auction-by-auction inverse bidding function of Guerre et al. (2000), vectorized over the $L_i$ auctions played by $i$.\footnote{To see this, recall that under Assumption 4 the $l$th element of $\Gamma_i(b_i|M)$ is simply the c.d.f. of the maximum bid among $i$’s rivals in auction $l$, evaluated at $b_{il}$. Hence the $l$th element of $\Upsilon_i(b_i|M)$ reduces to

$$\Upsilon_{i,l}(b_i|M) = b_{il} + \frac{\Gamma_{i,l}(b_{il}|M)}{\gamma_{i,l}(b_{il}|M)} \text{ where } \gamma_{i,l}(b_{il}|M) \equiv \frac{d\Gamma_{i,l}(b_{il}|M)}{db_{il}},$$

i.e., the usual standalone inverse bid function of Guerre et al. (2000) in auction $l$.}

The weights $\Psi_i(b_i|M)$ on $K_i$ in the latter correspond, intuitively, to the marginal effect of increasing each bid $b_{il}$ on $i$’s probability of winning higher-order combinations, relative to the marginal effect of $b_{il}$ on $i$’s probability of winning auction $l$.

Finally, observe that if the conjecture $\kappa_i(Z,W) = K_i$ is in fact correct, then we must have $V_i = \xi_i(b_i|M; K_i)$ almost surely. Hence to each candidate $K_i$ for $\kappa_i(Z,W_i)$, there corresponds a unique, identified candidate $\hat{F}_i(\cdot|M; K_i)$ for the unknown c.d.f. $F_i(\cdot|Z,X)$:

$$\hat{F}_i(v|M; K_i) = \int_{B_i} 1[\xi_i(B_i|M; K_i) \leq v] G_i(dB_i|M).$$  \hspace{1cm} (11)

In other words, if $\kappa_i(Z,W_i)$ were known, then we could recover $F_i(\cdot|X,Z)$ immediately through the identity $F_i(\cdot|Z,X) \equiv \hat{F}_i(\cdot|M; \kappa_i(Z,W_i))$. Identification of bidder $i$’s primitives therefore reduces to recovery of the unknown non-parametric function $\kappa_i(Z,W_i)$.
3.3 Nonparametric identification of complementarities based on variation in rival characteristics

In view of Proposition 1, it is also clear that further structure is necessary for identification: under Assumptions 1-4, we can identify valuations only up to complementarities. But suppose that, to these assumptions, we add the hypothesis that bidder $i$’s primitives $(F_i, \kappa_i)$ depend only on bidder $i$’s characteristics $Z_i$, not on the characteristics of rival bidders $Z_{-i}$:

**Assumption 5.** For all bidders $i$, $F_i(\cdot|Z, X) = F_i(\cdot|Z_i, X)$ and $\kappa_i(Z, W) = \kappa_i(Z_i, W)$.

Similar assumptions have been widely employed in the empirical auction literature; see, e.g., Guerre et al. (2009), and Somaini (2015) among others. We will show that under Assumption 5, variation in competitor characteristics $(Z_{-i}, W_{-i})$ induces a large (infinite) set of restrictions on the finite vector $\kappa_i(Z, W)$. Under mild conditions on variation in $(Z_{-i}, W_{-i})$ made precise below, these restrictions will have the unique solution $K_i = \kappa_i(Z, W)$, leading to nonparametric identification of $\kappa_i(Z, W)$ and hence the model as above.

Toward this end, consider any bidder $i = 1, ..., N$. Fix any realization of auction characteristics $X$, own characteristics $Z_i$, and combination characteristics $W_i$. Let $M \equiv (X, Z, W)$ and $M' \equiv (X', Z', W')$ be any two market structures such that the characteristics affecting $i$’s primitives are held constant, but the characteristics of $i$’s competitors (or factors affecting their complementarities) vary: that is, such that $X = X'$, $Z_i = Z_i'$, and $W_i = W_i'$, but either $Z_{-i} \neq Z_{-i}'$ or $W_{-i} \neq W_{-i}'$. In contrast to the simple two-auction illustration above, bids observed at market structures $M$ and $M'$ will now typically correspond to different realizations of $i$’s standalone valuations $V_i$. But, in view of Assumption 5, we know that $V_i$ is drawn from the same distribution $F_i(\cdot|X, Z_i)$ under both $M$ and $M'$. Furthermore, from Proposition 1, we know that for each market structure $M$ and each candidate complementarity $K_i$, there exists a unique, identified candidate $\tilde{F}_i(\cdot|M; K_i)$ for the unknown distribution $F_i(\cdot|X, Z_i)$. 
Hence, if \( K_i = \kappa_i(Z_i, W_i) \), then for almost every \( v \in \mathbb{R}^{L_i} \) we must have

\[
\hat{F}_i(v|M; K_i) = F_i(v|X, Z_i) = \hat{F}_i(v|M'; K_i).
\]

Clearly, if \( \hat{F}_i(\cdot|M; K_i) \) and \( \hat{F}_i(\cdot|M'; K_i) \) coincide almost everywhere, then the expectations of random vectors drawn from these distributions must also coincide. But recall that, by definition, \( \hat{F}_i(\cdot|M; K_i) \) is the distribution of the random vector \( \hat{V}_i \equiv \xi_i(B_i|M; K_i) \), where \( B_i \sim G_i(\cdot|M) \). Hence, if \( K_i = \kappa_i(Z_i, W_i) \), then in view of (12) we must also have

\[
\int_{B_i} \xi_i(B_i|M; K_i) G_i(dB_i|M) = \int_{B_i} \xi_i(B_i|M'; K_i) G_i(dB_i|M').
\]

Finally, recall that \( \xi_i(\cdot|M; K_i) \) is affine in \( K_i \). Hence we may equivalently rewrite each integral in (13) as an identified affine function of \( K_i \) as follows:

\[
\int_{B_i} \xi_i(B_i|M; K_i) G_i(dB_i|M) = \int_{B_i} \left[ \Upsilon_i(B_i|M) - \Psi_i(B_i|M) \cdot K_i \right] G_i(dB_i|M) 
\equiv \Upsilon_i(M) - \Psi_i(M) \cdot K_i,
\]

where \( \Upsilon_i(M) \), an identified \( L_i \times 1 \) vector, and \( \Psi_i(M) \), an identified \( L_i \times 2^{L_i} \) matrix, denote the expectations of the functions \( \Upsilon_i(\cdot|M) \) and \( \Psi_i(\cdot|M) \) defined in Proposition 1 with respect to bids drawn from \( i \)'s equilibrium bid distribution \( G_i(\cdot|M) \):

\[
\Upsilon_i(M) \equiv \int_{B_i} \Upsilon_i(B_i|M) G_i(dB_i|M), \\
\Psi_i(M) \equiv \int_{B_i} \Psi_i(B_i|M) G_i(dB_i|M).
\]

Substituting (14) into (13) under the hypothesis \( K_i = \kappa_i(Z_i, W_i) \), we ultimately obtain a system of \( L_i \) linear restrictions on the unknown vector \( \kappa_i(Z_i, W_i) \in \mathcal{K}_i \):

\[
[\Upsilon_i(M) - \Upsilon_i(M')] - [\Psi_i(M) - \Psi_i(M')] \cdot \kappa_i(Z_i, W_i) = 0.
\]
Recall that the first \( L_i + 1 \) elements of \( \kappa_i(Z_i, W_i) \) are zero by construction. Hence (15) is effectively a system of \( L_i \) equations in \( 2^{L_i} - L_i - 1 \) unknowns. When \( L_i > 2 \), we have 
\[ 2^{L_i} - L_i - 1 > L_i, \]
hence the system (15) alone will be insufficient to identify \( \kappa_i(Z_i, W_i) \).

But recall that (15) must hold for \textit{any} pair of markets \( M, M' \) such that \( X = X', Z_i = Z_i' \) and \( W_i = W_i' \). In other words, for given \((X, Z_i, W_i)\), every distinct realization of rival characteristics \((Z_{-i}, W_{-i})\) generates an additional set of \( L_i \) linear restrictions parallelling (15), all of which must hold simultaneously at \( K_i = \kappa_i(Z_i, W_i) \). Pooling such linear restrictions across many markets with varying rival characteristics \((Z_{-i}, W_{-i})\), we ultimately conclude:

**Proposition 2.** Consider any bidder \( i = 1, \ldots, N \) and any realization of \( i \)'s bidder and combination characteristics \((Z_i, W_i)\). Let \( M^0, M^1, \ldots, M^J \) be any collection of market structures such that \( Z_i, W_i \) and \( X \) are constant for all \( j = 1, \ldots, J \), and suppose that the submatrix formed by the last \( (2^{L_i} - L_i - 1) \) columns of the \( J(J - 1)L_i \times 2^{L_i} \) matrix 
\[
\Delta \bar{\Psi} \equiv \left[ \bar{\Psi}_i(M^j) - \bar{\Psi}_i(M^k) \right]_{j,k \in \{1, \ldots, J\}}
\]
has rank \( 2^{L_i} - L_i - 1 \). Then \( \kappa_i(Z_i, W_i) \) is identified.

Recall that the identification criterion (15) exploits only invariance of \textit{first moments} of \( F_i(\cdot|Z_i, X) \), whereas the underlying distributional invariance restriction (12) requires equality of \textit{all moments}. The system of equations in Proposition 2 merely provides a simple and testable sufficient condition under which the underlying system has a unique solution. Note also that variation in, e.g., number of rivals in each auction will produce exactly the kind of variation needed for full column rank of \( \Delta \bar{\Psi} \): nonlinear changes in probabilities of winning different combinations, which map into bidding as weights on the unknown vector \( \kappa_i(Z_i, W_i) \).

Even discrete variation in \( Z_{-i} \) thus naturally gives rise to full column rank of \( \Delta \bar{\Psi} \), yielding nonparametric identification of \( \kappa_i(Z_i, W_i) \) and hence the model as above.
3.4 Nonparametric identification of complementarities based on variation in characteristics of other objects

While the restriction that own primitives are invariant to competitor characteristics is both natural and widely employed, it could potentially be violated in environments with richer strategic interaction among players. For instance, if there is a competitive upstream market for sub-contractors, then capacity utilization by i’s rivals could in principle affect i’s costs. We therefore also consider nonparametric identification of $\kappa_i(\cdot)$ based on excludable variation in characteristics of other auctions. This key hypothesis underlying this approach, also widely maintained in the literature, is that standalone valuations in each auction $l$ depend only on object $l$’s characteristics $X_l$, not on the characteristics of other objects $X_{-l}$:

**Assumption 6.** For each bidder $i$ and object $l \in \mathcal{L}_i$, $F_{il}(\cdot|Z,X) = F_{il}(\cdot|Z,X_l)$.

This assumption allows both $F_i(\cdot|Z,X)$ and $\kappa_i(Z,W)$ to depend on $Z_{-i}$, but requires each marginal distribution to be invariant to characteristics of other objects. The subsequent identification argument closely follows the steps above, with variation in $X_{-l}$ replacing variation in $Z_{-i}$, and is therefore omitted for brevity.17 Obviously, where appropriate, Assumptions 5 and 6 can also be maintained jointly, as we do in our application.

3.5 Additional identifying restrictions induced by variation in characteristics of combinations

What about variation in combination characteristics $W_i$? In practice, $W_i$ is typically a function of observables $(X,Z)$. Nevertheless, it may be natural to consider variation in $W_i$ holding relevant features of $(X,Z)$ constant—for instance, by changing the distance between two projects, holding $i$’s distance to each project constant. Particularly when $\kappa_i(\cdot)$ is

---

17Note, however, that the rank condition in Proposition 2 follows less immediately from variation in $X_{-l}$ than from variation in $Z_{-i}$. For instance, if $X_l$ shifts all bidders’ valuations for auction $l$ by an identical additive quantity, then $X_l$ will simply shift all bids in auction $l$ additively, without changing $i$’s probability of winning any combination. Apart from this special case, however, variation in $X_{-l}$ will typically induce nonlinear variation in bids, and hence variation in $\Psi_i(\cdot)$ which is informative on $\kappa_i(\cdot)$.
assumed parametric, as will often be the case in practice, such variation can have considerable identifying power. Bearing caveats of functional dependence in mind, we thus briefly discuss additional identifying restrictions induced by variation in \( W_i \).

Toward this end, recall from Proposition 1 that we must have \( V_i = \Upsilon_i(b_i|M) - \Psi_i(b_i|M) \cdot \kappa_i(Z,W_i) \) almost surely. This in turn implies the equilibrium identity

\[
E[V_i|X,Z] = \bar{\Upsilon}_i(M) - \bar{\Psi}_i(M) \cdot \kappa_i(Z,W_i).
\] (16)

Suppose that element \( W_{ik} \) of \( W_j \) varies conditional on \((X,Z)\). Assume for simplicity that the right-hand side of (16) is differentiable in \( W_{ik} \); otherwise, we may proceed in finite differences as above. Differentiating both sides of (16) with respect to \( W_{ik} \), noting that \( E[V_i|X,Z] \) does not depend on \( W_{ik} \), and rearranging, we obtain

\[
\bar{\Psi}_i(M) \cdot \frac{\partial \kappa_i(Z,W_i)}{\partial W_{ik}} = \frac{\partial \Upsilon_i(M)}{\partial W_{ik}} - \frac{\partial \Psi_i(M)}{\partial W_{ik}} \cdot \kappa_i(Z,W_i).
\] (17)

Equation (17) defines a further set of restrictions on the relationship between changes in \( \kappa_i(Z,W_i) \) and levels of \( \kappa_i(Z,W_i) \). Depending on the scope of variation in \( W_i \) conditional on \((X,Z)\), these may not be sufficient in themselves to identify \( \kappa_i(Z,W_i) \). But if \( \kappa_i(Z,W_i) \) is assumed parametric, so that levels and changes in \( \kappa_i(Z,W_i) \) both depend on some underlying parameter \( \theta \), then equations of the form (17) will convey substantial information on this parameter. We return to this point in our empirical application below.

### 3.6 Extensions

While our analysis so far has focused on the baseline model defined in Assumptions 1-4, our identification insights extend to accommodate endogenous participation, unobserved heterogeneity, richer complementarities, and potential mass points in bids. We develop these extensions in detail in Appendices B-E below. For completeness, however, we also describe the main ideas of each extension briefly here.
Appendix B formally embeds our bid-stage analysis within a two-stage entry and bidding game. In this extension, following Levin and Smith (1994), Krasnokutskaya and Seim (2011), Moreno and Wooders (2011), Athey et al. (2011), Groeger (2014), and Li and Zhang (2015) among others, we interpret entry as a process of value discovery. Bidders first simultaneously choose which combinations of auctions to enter on the basis of the common-knowledge primitives \((F_i, K_i)_{i=1}^N\) plus a vector of private, potentially combinatorial, entry costs. Conditional on entry, each bidder \(i\) then discovers their standalone valuations \(V_i\) for each auction in which they have entered. Finally, based on private valuations \(V_i\) plus the common-knowledge characteristics \(M_i\), entering bidders submit bids as above.

How does such endogenous participation change our understanding of the underlying bidding game? Clearly, the sets of auctions which bidders actually enter will not be random; rather, bidders will endogenously select into both auctions for which they expect high valuations and combinations for which they anticipate positive complementarities. This implies, for instance, that the complementarities we actually observe will differ from those which would arise if bidders were randomly assigned to auctions, a point to which we return in interpreting our results. Crucially, however, so long as this selection is solely on the basis of the common-knowledge primitives \((F_i, K_i)_{i=1}^N\), entry in fact strengthens prospects for identification in two respects. First, this model of participation provides a formal equilibrium justification for Assumption 5 above. Second, the combinations of auctions which bidders choose to enter will themselves convey additional information about complementarities. We develop both points more fully in Appendix B.

Next, in Appendix C, we extend our identification analysis to accommodate unobserved

---

18 If bidders instead select into entry on the basis of private information about their valuations \(V_i\), as in the “selective entry” models of Roberts and Sweeting (2013) and Gentry and Li (2014), then matters would be more complicated. In this case, for instance, the set of potential competitors bidder \(i\) faces will generally affect the information sets at which bidder \(i\) choose to enter, and hence the distribution of valuations \(F_i\) drawn upon entry. Even in this case, however, the identification insights below still apply so long as bidder \(i\) observes the set of rival entrants prior to bidding. In this case, one would include the set of potential competitors in each auction in \(X\), and the set of actual entrants in \(Z\). Furthermore, regardless of whether bidders observe actual rivals prior to bidding, the testing insights we develop below continue to apply: even when entry is selective, cross-auction spillovers can arise only if auctions are not additively separable.
auction heterogeneity, modeled as an auction-level characteristic $A_l$ such that $V_{il} = U_{il} + A_l$, with $U_i \equiv [U_{il}]^{L_i}_{l=1}$ independent private information for each bidder and $A_l$ common knowledge to bidders but not the econometrician. The main complication induced by such unobserved heterogeneity is a first step in which, following Krasnokutskaya (2011), one nets out variation in bids driven by the separable unobservables $A \equiv [A_l]^{L}_l$. Non-parametric identification of bidder-level primitives then proceeds as described above.

Third, in Appendix D, we generalize our non-parametric identification results to the case where complementarities are stochastic but their randomness can be fully explained by the standalone valuations. Such a case could arise if, for instance, winning two auctions together increases i’s valuation for one or both objects by a fixed percentage.

Finally, in Appendix E, we extend our analysis to accommodate potential violations of the regularity conditions maintained in Assumption 4 above, such as might arise if bid spaces are discrete or if bidders submit bids which never win. In this case, the arguments above deliver (at worst) partial, rather than point, identification of complementarities.

4 Application: Michigan Highway Procurement

We now turn to our empirical application: Michigan Department of Transportation (MDOT) highway construction and maintenance contracts. MDOT allocates contracts for a wide range of highway construction and maintenance services via low-price sealed-bid auctions. The vast majority of MDOT projects are allocated via large simultaneous letting rounds, which take place on average every three weeks. There are an average of 45 auctions per letting round and more than half (56 percent) of bidders submit bids on multiple contracts within a letting. A bid is an itemized description of unit costs for each line item specified in contract plans; bids are submitted to MDOT project by project, with the winner of each project the

---

19There are only two months without lettings.
20MDOT runs a pre-qualification process, which ensures quality of work. The process involves a check on the financial status of the firm and its backlogs from all construction activities. A bid submission includes a detailed break down of all costs involved in the contract. The winner is determined solely by the total cost of the project.
bidder submitting the bid involving the lowest total project costs. Contracts are advertised up to ten weeks prior to letting, with the closing deadline for submitting, amending or withdrawing bids typically 10am on the letting date. MDOT then publically opens bids and allocates contracts, with winning bidders held liable for completion of contracts won. In view of prior evidence on complementarities and capacity constraints in highway procurement, we expect factors such as capacity constraints, project proximity, project types, and scheduling overlap to induce substantial non-additivities in bidder payoffs across auctions.

4.1 Data

MDOT provides detailed records on contracts auctioned, bids received, and letting outcomes on its letting website (http://www.michigan.gov/mdot). Drawing from these records, we observe data on (almost) all contracts auctioned by MDOT over the sample period January 2005 to March 2014.21 Our sample includes a total of 8224 auctions, where for each auction the following information is observed: project description, project location, pre-qualification requirements, the internal MDOT engineer’s estimate of total project cost, and the list of participating firms and their bids. Based on project descriptions, we classify projects into five project types: bridge work, major construction, paving (primarily hot-mix asphalt), safety (e.g. signing and signals), and miscellaneous, leading to a final distribution of projects across types summarized in Table 1. As evident from Table 1, roughly 80 percent of contracts are for road and bridge construction and maintenance broadly defined, with the remainder split between safety and other miscellaneous construction.

The data contains information on a total of 859 unique bidders active in the MDOT marketplace over our sample period, which we classify by size and scope of activity as follows. We define a bidder as “regular” if it submitted more than 100 bids in the sample period, and “fringe” otherwise. This yields a total of 36 regular bidders, with all other bidders classified

\[21\] MDOT records for a small number of contracts are incomplete. Although we have data from October 2002 to March 2014, we have discarded the first few years (from October 2002 to December 2004) as we use lettings from these years to construct bidder backlog variables.
Table 1: Summary of Projects by Type

<table>
<thead>
<tr>
<th>Contract Type</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bridge</td>
<td>13.33</td>
</tr>
<tr>
<td>Major Construction</td>
<td>9.64</td>
</tr>
<tr>
<td>Paving</td>
<td>56.33</td>
</tr>
<tr>
<td>Safety</td>
<td>12.25</td>
</tr>
<tr>
<td>Miscellaneous</td>
<td>8.45</td>
</tr>
</tbody>
</table>

as “fringe.” For the subsample of bidders who have submitted more than 50 bids, we also collect data on number and location of plants by firm. This data is derived from a variety of sources: OneSource North America Business Browser, Dun and Bradstreet, Hoover’s, Yellowpages.com and firms’ websites. We then further classify bidders as “large” or “small” based on this data, with “large” bidders those owning at least 9 plants in Michigan. We thus obtain a final classification of 3 large regular bidders, 33 small regular bidders, and 823 fringe bidders (of which 2 are large bidders) in the MDOT marketplace.

Table 2 surveys the auction side of the MDOT marketplace. The first key feature emerging from this table is the large number of contracts auctioned simultaneously in the market: a mean of 45 per letting, with a maximum of 133 on a single letting date.\(^{22}\) On average about five bids are received per contract, which is small relative to the average number of bidders (approximately 84) active in any given letting. For each contract, MDOT prepares an internal “Engineer’s Estimate” of expected procurement cost released to bidders before bidding; as evident from the dispersion in this estimate, projects vary substantially in size and complexity. The statistic “Money Left on the Table” measures the percent difference between lowest and second-lowest bids. On average, this is 7.4 percent, or roughly $112,000 per contract, suggesting the presence of substantial uncertainty over rival bids.

Table 3 summarizes bidder behavior in the MDOT marketplace. Consistent with Table 2, the average bidder competes in roughly 2.7 auctions per round, with large and regular bidders competing in substantially more. The variable “backlog” provides a bidder-specific measure

\(^{22}\)Note that smaller supplemental lettings are occasionally held two or three weeks after the main letting in a given month.
of capacity utilization. As usual, we define backlog for bidder $i$ at date $t$ as the sum of work remaining among projects $l$ won by $i$ up to $t$, where work remaining on project $l$ at date $t$ is defined as total project size (measured by the engineer’s estimate) times the proportion of scheduled project days remaining at date $t$. Note that number of bids submitted by any given bidder is small relative to the number of auctions in the marketplace, with even large bidders competing in less than fifteen percent of total auctions on average.

Finally, Figure 1 plots the histogram (over all bidders $i$ and lettings $t$) of the number of bids submitted by bidder $i$ in letting $t$. As evident from Figure 1, more than 55 percent of active bidders submit multiple bids in the same letting. Despite this, it is relatively uncommon for a typical bidder to compete in a large number of auctions; roughly 92 percent of bidders in our sample bid in 6 or fewer auctions and only 2.5 percent bid in more than 10.

---

23An observation for the purposes of Figure 1 is thus a bidder-letting pair.
4.2 Descriptive regressions

We next explore a series of simple regressions designed to explore the economic implications of simultaneous bidding in the MDOT marketplace. The unit of analysis in these regressions is a bidder-auction-round combination. The dependent variable is the log of bid submitted by bidder $i$ in auction $l$ in letting $t$, regressed on a vector of covariates intended to capture the effect of own- and cross-auction characteristics on $i$'s bid in auction $l$.

**Regression specification**  As usual, we control for a number of auction-level characteristics which we expect to be key direct determinants of $i$'s bid in auction $l$: the size of auction $l$, proxied by the MDOT engineer’s estimate of project cost, the level of competition $i$ faces in auction $l$, and the distance between project $l$ and $i$'s base of operations.$^{24}$ To control for the direct cost effects of capacity usage, we also include a standardized bidder-level backlog variable, derived from the backlog measure described above by subtracting the mean and

---

$^{24}$We construct for each bidder-project pair the minimum straight-line distance (in miles) between any of $i$’s plants and the centroid of the county in which project $l$ is located. We take the shortest distance if bidder $i$ owns multiple plants.
dividing by the standard deviation of backlogs for each bidder over time.

To explore cross-auction interaction in the MDOT marketplace, we seek a set of covariates relevant for combination payoffs but irrelevant for standalone valuations after conditioning on characteristics of auction $l$. Toward this end, we construct the following covariates.

To control for cross-auction competition which may shift combination win probabilities, we consider the total number of rivals across all auctions played by bidder $i$. The effect of cross-auction competition on $i$’s bids in auction $l$ is theoretically ambiguous, depending both on the sign of $\kappa_i$ and on strategic responses by rival bidders. Heuristically, however, if objects are substitutes, we expect greater competition in auction $k$ to increase marginal returns to winning auction $l$, and conversely if objects are complements.

To capture the presence of capacity constraints or diseconomies of scale, we consider two variables. First, as a direct measure of total project size, we consider the (log of) the sum of engineer’s estimates across all auctions in which $i$ is bidding. Second, as a measure of the degree of schedule overlap on projects for which $i$ is bidding, we consider the total number of overlapping days for projects for which $i$ submits bids, scaled by the sum of days scheduled for each of these projects. Insofar as marginal costs are increasing in capacity utilization, we expect the coefficients on these variables to be positive.

In principle, complementarities arising between similar projects may differ from those arising between different projects. To account for this possibility, we consider an index of concentration for the types of projects for which $i$ is bidding, defined as a Herfindahl index over shares of each project type in $i$’s participation set. A negative coefficient on this index is interpreted as a relative complementarity between similar projects.

Finally, as a proxy for potential economies or diseconomies induced by distance between projects, we consider the (log of) total distance between the current project and each other project for which $i$ bids, normalized by the total distance between each of these projects and the closest plant owned by bidder $i$. Insofar as relatively more distant projects potentially reduce economies of scale, we expect this variable to have a positive sign.
Regression results  Table 4 reports OLS estimates for our baseline regression specifications: log bids on the own- and cross-auction characteristics defined above. All regression specifications include a full set of bidder type, project type, and letting date indicators, with standard errors clustered by bidder and round to allow for correlation in elements of $b_{it}$.

Estimated effects of own-auction characteristics correspond closely both to our prior and to findings elsewhere in the literature. As expected, bids are increasing almost one for one in project size, with the coefficient on log engineer’s estimate exceeding 0.97. Bidders facing more competition bid more aggressively, with one additional competitor associated with a 4 to 5 percent decrease in average bids. Finally, a one percent increase in $i$’s distance to the project leads to about a 2 percent increase in $i$’s bid on average.

More importantly, estimated cross-auction effects are also highly significant, with magnitudes stable across specifications and signs broadly consistent with our prior expectations. The positive coefficient on log sum of engineer’s estimates suggests that competing for many large projects leads to a substantial decrease in aggressiveness by bidder $i$ in auction $l$, with the negative coefficient on same-type projects suggesting that this effect is ameliorated slightly when the two projects are of the same type. Similarly, the positive sign on log distance among projects suggests that increasing distance to other projects reduces the synergies among them. Finally, the significant negative coefficient on total number of rivals in auctions participated by $i$ suggests that facing more competition across auctions leads bidder $i$ to bid more aggressively in auction $l$. Taken together, these results corroborate the hypothesis that simultaneous bidding induces strategic spillovers across auctions.

5 Structural estimation of complementarities

We now turn to this paper’s primary interest: structural estimation of the function $\kappa_i(\cdot)$ describing preferences over combinations. In principle, the results in Section 3 support fully non-parametric estimation of $\kappa_i$. In practice, of course, the dimensionality of the problem
Table 4: OLS Estimates of Cross-Auction Effects

\[ y = \ln(bid) \]

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Log engineer’s estimate</td>
<td>0.971***</td>
<td>0.9751***</td>
</tr>
<tr>
<td></td>
<td>(0.0021)</td>
<td>(0.0018)</td>
</tr>
<tr>
<td>Log number of rivals</td>
<td>-0.0509***</td>
<td>-0.0472***</td>
</tr>
<tr>
<td></td>
<td>(0.0079)</td>
<td>(0.0047)</td>
</tr>
<tr>
<td>Log distance to project</td>
<td>0.021***</td>
<td>0.0137***</td>
</tr>
<tr>
<td></td>
<td>(0.0019)</td>
<td>(0.0011)</td>
</tr>
<tr>
<td>Log days to project start</td>
<td>0.0038***</td>
<td>0.0034**</td>
</tr>
<tr>
<td></td>
<td>(0.0012)</td>
<td>(0.0017)</td>
</tr>
<tr>
<td>Standardized backlog</td>
<td>0.0028**</td>
<td>0.0032***</td>
</tr>
<tr>
<td></td>
<td>(0.0012)</td>
<td>(0.0009)</td>
</tr>
<tr>
<td>Log number of big rivals faced</td>
<td>-0.0018</td>
<td>-0.0286***</td>
</tr>
<tr>
<td></td>
<td>(0.0055)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>Log number of regular rivals faced</td>
<td>0.0279***</td>
<td>0.0372***</td>
</tr>
<tr>
<td></td>
<td>(0.0052)</td>
<td>(0.0043)</td>
</tr>
<tr>
<td>Multiple-bid indicator</td>
<td>-0.094***</td>
<td>-0.1741***</td>
</tr>
<tr>
<td></td>
<td>(0.0243)</td>
<td>(0.0256)</td>
</tr>
<tr>
<td>Log sum engineer’s estimate across played auctions</td>
<td>0.0062***</td>
<td>0.0117***</td>
</tr>
<tr>
<td></td>
<td>(0.0017)</td>
<td>(0.0017)</td>
</tr>
<tr>
<td>Log sum number of rivals across played auctions</td>
<td>-0.0161***</td>
<td>-0.0146***</td>
</tr>
<tr>
<td></td>
<td>(0.0033)</td>
<td>(0.0027)</td>
</tr>
<tr>
<td>Log distance across played projects</td>
<td>0.0041**</td>
<td>0.0099***</td>
</tr>
<tr>
<td></td>
<td>(0.0021)</td>
<td>(0.002)</td>
</tr>
<tr>
<td>Fraction overlapping time across projects</td>
<td>0.0174***</td>
<td>0.0129**</td>
</tr>
<tr>
<td></td>
<td>(0.0043)</td>
<td>(0.0055)</td>
</tr>
<tr>
<td>Same-type-auctions concentration index</td>
<td>-0.0099**</td>
<td>-0.0265***</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
<td>(0.0058)</td>
</tr>
<tr>
<td>Big bidder</td>
<td>-</td>
<td>-0.0258***</td>
</tr>
<tr>
<td></td>
<td>- (0.005)</td>
<td></td>
</tr>
<tr>
<td>Regular Bidder</td>
<td>-</td>
<td>-0.003</td>
</tr>
<tr>
<td></td>
<td>- (0.0023)</td>
<td></td>
</tr>
<tr>
<td>Year FE, Month FE, Auction type FE</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>Bidder type FE</td>
<td>NO</td>
<td>YES</td>
</tr>
<tr>
<td>Bidder ID FE</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>R-squared</td>
<td>98.97</td>
<td>97.94</td>
</tr>
</tbody>
</table>

Unit of analysis is bidder-auction-round, with standard errors clustered by bidder within each round. There are 40,624 observations. Variables log of engineer’s estimate, log of number of rivals in the auction and log of distance to the county centroid measure size, strength of competition, and distance to project respectively. Remaining variables proxy for cross-auction characteristics: number of rivals in other auctions, sum engineer’s estimate, distance to auctions scaled by distance to project i in which i is competing and number of overlapping days among projects scaled by the total number of days to completion.
renders this infeasible. We therefore implement our structural procedure in two steps. First, following Cantillon and Pesendorfer (2006) and Athey et al. (2011) among others, we estimate a parametric approximation to the equilibrium distribution $G_{it}$ of bids submitted by each bidder $i$ in letting $t$. Second, we map these estimates through the inverse bidding function (8) to obtain a set of moment conditions based on the exclusion restrictions discussed in Section 3, which we then use to estimate parameters in $\kappa_i$. Following Groeger (2014), we assume there is no binding reserve price.  

5.1 First step: estimation of $G_{it}$

As usual, the first step in our procedure is to estimate the conditional joint distribution $G_{it}$ and pdf $g_{it}$ of bids submitted by each bidder $i$ in letting $t$. In view of the dimensionality of this problem, we follow Cantillon and Pesendorfer (2006) and Athey et al. (2011) in estimating a parametric approximation to this joint distribution, which we specify as follows. We model the $L_{it} \times 1$ bid vector $b_{it}$ as drawn from a multivariate log-normal distribution characterized by mean vector $\mu_{it}$ and variance-covariance matrix $\Sigma_{it}$:

$$\ln(b_{it}) \sim g(\mu_{it}, \Sigma_{it}).$$

We allow the parameters $\mu_{it}$ and $\Sigma_{it}$ to depend on a vector of observables including $i$’s characteristics $Z_{it}$, project characteristics $X_{it}$, combination characteristics $W_{it}$, and the number and types of rivals $i$ faces within and across auctions. Specifically, for each auction $l = 1, ..., L_{it}$ played by $i$, we model the mean and variance of $\ln(b_{it,l})$ as $\mu_{it,l} = \alpha \cdot M^{\mu}_{it,l}$ and $\sigma_{it,l}^2 = \exp(\beta \cdot M^{\sigma}_{it,l})$ respectively, where $M^{\mu}_{it,l}$ and $M^{\sigma}_{it,l}$ are vectors of covariates specified in Panels A and B of Table 5, and $\alpha$ and $\beta$ are parameter vectors to be estimated. Meanwhile,

---

25When a bidder is the sole participant (which happens only 137 times out of 8824 auction analyzed), he will face MDOT that draws a completion cost from a fringe bidder’s cost.
we model the covariance $\rho_{it,kl}$ between distinct elements $\ln(b_{it,k})$ and $\ln(b_{it,l})$ of $\ln(b_{it})$ as

$$
\rho_{it,kl} = \frac{\exp(\gamma \cdot M^\rho_{it,kl} - 1)}{\exp(\gamma \cdot M^\rho_{it,kl} + 1)},
$$

where $M^\rho_{it,kl}$ is a vector of interactions between observable characteristics of projects $k$ and $l$ specified in Panel C of Table 5, and $\gamma$ is a vector of parameters to be estimated.

For computational reasons, we restrict attention to bidders who compete in 16 or fewer auctions, which represents approximately 99.5 percent of all bidders. We estimate the model above by maximum likelihood on this subsample, pooling data from bidders that participate in different numbers of auctions. The log-likelihood function is therefore equal to:

$$
\sum_{t=1}^{T} \sum_{i=1}^{N_t} \log g(b_{it}|\mu_{it}, \Sigma_{it}).
$$

Table 5 reports maximum likelihood estimates of the first-step parameters ($\alpha, \beta, \gamma$) determining the distribution $g(\mu_{it}, \Sigma_{it})$. In Panel A, we report coefficient estimates $\hat{\alpha}$ on covariates $M^\mu_{it,l}$ appearing in the mean function $\mu_{it}$; not surprisingly, these are very similar to coefficients in our descriptive regressions. Panel B reports coefficients $\hat{\beta}$ on covariates $M^\sigma_{it,l}$ in the variance function $\sigma^2_{it,l}$, which suggest that bidders competing in multiple auctions and for larger projects submit less dispersed bids. Finally, in Panel C, we report coefficients $\hat{\gamma}$ on covariates $M^\rho_{it,kl}$ in the covariance function $\rho_{it,kl}$. These suggest at least two broad patterns in bidding behavior across auctions. First, bidders bid tend to bid more similarly for projects in the same county or of the same type. Second, when competing for projects whose schedules overlap, bidders tend to bid for one more aggressively than the other. This is consistent with our prior that overlapping schedules exacerbate diseconomies of scale.

To evaluate goodness of fit of this first-step model, Figure 2 plots the observed distribution of log bids across all auctions and bidders, together with predicted distribution of log bids

\[\text{While the parametrization of } \Sigma_{it} \text{ does not imply its positive semi-definitiveness, the estimated variance-covariance matrix is positive semi-definite.}\]
Table 5: First-step MLE estimates of parameters in $G_i$

<table>
<thead>
<tr>
<th></th>
<th>Mean $\mu_{ilt}$</th>
<th>$\hat{\alpha}$</th>
<th>MLE SEs</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Auction $l$ and bidder characteristics</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>0.3432</td>
<td>0.0159</td>
<td>0.312</td>
<td>0.3744</td>
</tr>
<tr>
<td>Log engineer’s estimate</td>
<td>0.98</td>
<td>0.0009</td>
<td>0.9782</td>
<td>0.9818</td>
</tr>
<tr>
<td>Log distance to project</td>
<td>0.0148</td>
<td>0.0009</td>
<td>0.013</td>
<td>0.0166</td>
</tr>
<tr>
<td>Log rivals in auction</td>
<td>-0.0478</td>
<td>0.0027</td>
<td>-0.0531</td>
<td>-0.0425</td>
</tr>
<tr>
<td>Log days to the start</td>
<td>0.0032</td>
<td>0.0008</td>
<td>0.0016</td>
<td>0.0048</td>
</tr>
<tr>
<td>Standardize backlog</td>
<td>0.0035</td>
<td>0.001</td>
<td>0.0015</td>
<td>0.0055</td>
</tr>
<tr>
<td>Big bidder</td>
<td>-0.0233</td>
<td>0.0052</td>
<td>-0.0335</td>
<td>-0.0131</td>
</tr>
<tr>
<td>Regular bidder</td>
<td>-0.0039</td>
<td>0.0025</td>
<td>-0.0088</td>
<td>0.001</td>
</tr>
<tr>
<td>Log number of big rivals faced</td>
<td>-0.0233</td>
<td>0.0031</td>
<td>-0.0294</td>
<td>-0.0172</td>
</tr>
<tr>
<td>Log number of regular rivals faced</td>
<td>0.0305</td>
<td>0.0023</td>
<td>0.026</td>
<td>0.035</td>
</tr>
<tr>
<td>Bidder Type FE</td>
<td>YES</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Auction Type FE</td>
<td>YES</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Other auctions characteristics</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Multiple bids dummy</td>
<td>-0.13</td>
<td>0.0206</td>
<td>-0.1704</td>
<td>-0.0896</td>
</tr>
<tr>
<td>Same-type-auctions index</td>
<td>-0.0215</td>
<td>0.005</td>
<td>-0.0313</td>
<td>-0.0117</td>
</tr>
<tr>
<td>Fraction overlapping time</td>
<td>0.0177</td>
<td>0.0035</td>
<td>0.0108</td>
<td>0.0246</td>
</tr>
<tr>
<td>Log sum engineer’s (across $l$)</td>
<td>0.008</td>
<td>0.0014</td>
<td>0.0053</td>
<td>0.0107</td>
</tr>
<tr>
<td>Log sum rivals (across $l$)</td>
<td>-0.0103</td>
<td>0.0022</td>
<td>-0.0146</td>
<td>-0.006</td>
</tr>
<tr>
<td>Log distance across played projects</td>
<td>0.0113</td>
<td>0.0017</td>
<td>0.008</td>
<td>0.0146</td>
</tr>
<tr>
<td>Year FE</td>
<td>YES</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Month FE</td>
<td>YES</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Variance $\sigma_{ilt}^2$</th>
<th>$\hat{\beta}$</th>
<th>MLE SEs</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.1013</td>
<td>0.0742</td>
<td>-0.0441</td>
<td>0.2467</td>
</tr>
<tr>
<td>Multiple bids dummy</td>
<td>-0.2108</td>
<td>0.019</td>
<td>-0.2480</td>
<td>-0.1736</td>
</tr>
<tr>
<td>Log engineer’s estimate</td>
<td>-0.2647</td>
<td>0.0054</td>
<td>-0.2753</td>
<td>-0.2541</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Covariance $\rho_{ilt}$</th>
<th>$\hat{\gamma}$</th>
<th>MLE SEs</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.1948</td>
<td>0.0231</td>
<td>0.1495</td>
<td>0.24</td>
</tr>
<tr>
<td>Same county projects</td>
<td>0.2213</td>
<td>0.0275</td>
<td>0.167</td>
<td>0.2752</td>
</tr>
<tr>
<td>Same type projects</td>
<td>0.1285</td>
<td>0.0189</td>
<td>0.0915</td>
<td>0.1655</td>
</tr>
<tr>
<td>Fraction overlapping time</td>
<td>-0.0261</td>
<td>0.0203</td>
<td>-0.0661</td>
<td>0.0139</td>
</tr>
</tbody>
</table>
implied by the estimates in Table 5. As can be seen in Figure 2, the fit of our parametric approximation appears excellent, reinforcing confidence in the first-step estimates above.

5.2 Second step: estimation of complementarities

In view of our low-bid procurement application, we here reinterpret the general model in Section 2 as follows. Let $V_{it,l}$ be $i$’s private standalone cost for completing project $l \in \mathcal{L}_{it}$, and $\kappa_i(Z_{it}, W_{it})$ be the vector of cost complementarities $i$ associates with each combination $\omega \in \Omega_{it}$, where $\kappa_i^\omega(Z_{it}, W_{it}) \geq 0$ implies higher completion costs.
We model the complementarity \( i \) associates with combination \( \omega \) as linear in a \( 1 \times Q \) vector of bidder and combination-level observables \( M_{\omega}^{it} \), including factors such as total size, distance among projects, overlap between projects, Herfindahl index of project types in combination \( \omega \), and interactions of these, together with a set of dummies for size and type of bidder \( i \):

\[
\kappa_{i}^{\omega}(Z_{it}, W_{it}) = M_{\omega}^{it} \theta_{0},
\]

where \( \theta_{0} \subset \Theta \) is a \( Q \times 1 \) vector of structural parameters to be estimated. With slight abuse of notation, let \( M_{\omega}^{it} \) be the \( 2^{L_{it}} \times Q \) matrix whose rows collect covariate vectors \( M_{\omega}^{it} \) describing each combination \( \omega \in \Omega_{it} \). By construction, under (18), we then have \( \kappa_{i}(Z_{it}, W_{it}) = M_{\omega}^{it} \theta_{0} \).

We consider estimation maintaining both Assumptions 5 and 6, leveraging variation in rival characteristics \( Z_{-it} \), characteristics of other auctions \( X_{-lt} \), and combination characteristics \( W_{it} \) to identify \( \theta_{0} \). The essence of our estimation strategy is to exploit invariance of first moments of \( V_{it,l} \) to the excluded variables \( (Z_{-it}, X_{-lt}, W_{it}) \). Specifically, adapting arguments in Section 3, under Assumptions 5 and 6 we must have

\[
E[V_{it,l} | X_{lt}, Z_{it}] = E_{B_{i}}[\Upsilon_{i}(B_{i} | M_{t}) - \Psi_{i}(B_{i} | M_{t}) \cdot \kappa_{i}(Z_{it}, W_{it}) | M_{t}].
\]  

In practice, we specify the conditional mean \( E[V_{it,l} | X_{lt}, Z_{it}] \) as a linear index of the characteristics of bidder \( i \) and auction \( l \): \( E[V_{it,l} | X_{lt}, Z_{it}] = M_{\alpha}^{it} \theta_{1} \), where \( M_{\alpha}^{it} \equiv [X_{lt}, Z_{it}] \) collects auction- and bidder-level covariates, and \( \theta_{1} \) are parameters to be estimated. In view of the parametric form (18) for \( \kappa_{i}(Z_{it}, W_{it}) \), the equilibrium restriction (19) thus simplifies to:

\[
E_{B_{i}}[\Upsilon_{i}(B_{i} | M_{t}) - \Psi_{i}(B_{i} | M_{t}) \cdot M_{\omega}^{it} \theta_{0} - M_{\alpha}^{it} \theta_{1} | M_{t}] = 0.
\]  

As usual, we translate the conditional moment restrictions (20) into unconditional moment restrictions by selecting instruments, \( I_{it} \), from market characteristics \( M_{t} \). We then estimate
the unknown parameters $\theta \equiv (\theta_0, \theta_1)$ using GMM based on the moment restrictions

$$E[(\Upsilon_i(B_i|M_t) - \Psi_i(B_i|M_t) \cdot M^s_{it}\theta_0 - M^v_{it}\theta_1)\prime I_{it}] = 0. \quad (21)$$

In practice, we implement estimation based on these moment conditions as follows. To minimize the computational costs associated with constructing combinatorial win probabilities, we focus on the 92 percent of bidders competing in six or fewer auctions. For each bid vector $b_{it}$ in this estimation sample, we first construct empirical analogs $\hat{\Upsilon}_{it}$ and $\hat{\Psi}_{it}$ to the equilibrium objects $\Upsilon_i(b_{it}|M_t)$ and $\Psi_i(b_{it}|M_t)$:

$$\hat{\Upsilon}_{it} \equiv b_{it} + \nabla \hat{\Gamma}_i(b_{it}|M_t)^{-1}\hat{\Gamma}_i(b_{it}|M_t)$$
$$\hat{\Psi}_{it} \equiv \nabla \hat{\Gamma}_i(b_{it}|M_t)^{-1}\nabla \hat{P}_i(b_{it}|M_t),$$

where estimates $\hat{\Gamma}_i(b_{it}|M_t)$, $\hat{P}_i(b_{it}|M_t)$ for $\Gamma_i(b_{it}|M_t)$, $P_i(b_{it}|M_t)$ are derived directly from our first-step estimates $(\hat{G}_i(\cdot|M_t))_{i=1}^{N_t}$ of equilibrium bid distributions, and we approximate gradients with finite differences. We then define the structural residual

$$\hat{\epsilon}_{it} \equiv \hat{\Upsilon}_{it} - \hat{\Psi}_{it} \cdot M^s_{it}\theta_0 - M^v_{it}\theta_1. \quad (22)$$

Finally, letting $\theta = (\theta_0, \theta_1)$, we construct a moment function for bidder $i$ at time $t$ by averaging interactions of instruments and residuals across auctions played by $i$:

$$m_{it}(\theta) = \frac{1}{L_{it}} \sum_{i=1}^{L_{it}} I_{itt} \hat{\epsilon}_{itt}. \quad (23)$$

Our estimator $\hat{\theta}$ for $\theta$ is then obtained as the solution to the GMM criterion

$$\hat{\theta} = \arg\min_{\theta} \left( \sum_{t=1}^{T} \sum_{i=1}^{N_t} m_{it}(\theta) \right)^\prime W^{-1} \left( \sum_{t=1}^{T} \sum_{i=1}^{N_t} m_{it}(\theta) \right). \quad (24)$$
Specifically, we consider an iterative GMM procedure in which the initial weight matrix $W^0$ is set equal to the identity matrix, and then the estimated coefficients and the covariance matrix are alternatively updated until convergence.\textsuperscript{27} We cluster residuals at the bidder level within each letting, accounting for first-step estimation error via bootstrapping.

5.3 The main result: structural estimates of $\theta_0$

Table 6 reports the main structural estimates as outlined above. Combination-level variables have been scaled by the sum of engineer’s estimates among projects in the combination, so that coefficients are interpreted as percentage cost changes induced by a combination win, relative to the total size of the combination won. Recalling that negative signs reflect “positive complementarities” (lower costs) while positive signs reflect “negative complementarities” (higher costs), these coefficients have the following economic interpretations.

The variable “Sum of engineer’s estimates” reflects the total size of projects in a combination, with a positive coefficient suggesting that more total work renders a joint win less valuable, as we would expect in the presence of capacity constraints. The variables “Fraction overlapping time” and “Fraction overlapping time $\times$ Sum engineer’s estimates” measure the effect of schedule overlap on costs, with coefficients suggesting that perfect schedule overlap increases the average completion costs by about 7.5% of combination size, slightly decreasing as size increases. Meanwhile, the coefficient on “Same-type auction index” suggests that more homogeneous combinations are more appealing; a 0.1 change in the Herfindahl index of project types reduces costs by 9.5% of combination size. Finally, big and regular bidders seem to have larger cost synergies, although these effects are not significant at the 5% level.

With the exception of the coefficient on distance between projects, which is negative al-

\textsuperscript{27}That is, starting from $W^0$ equal to the identity matrix, we obtain $\theta^k$ by minimizing (24) at $W = W^{k-1}$. We then construct a new weighting matrix $W^k$ by

$$W^k = \frac{1}{\sum_{t=1}^{T} \sum_{i=1}^{N_t} m_i(\hat{\theta}^{k-1})m_i(\hat{\theta}^{k-1})'},$$

and minimize (24) to obtain a new estimate $\theta^{k+1}$, iterating until estimates $\theta^k, \theta^{k+1}$ differ by less than $10^{-6}$. 

40
Table 6: Estimated complementarity parameters $\theta_0$

<table>
<thead>
<tr>
<th>Combination characteristics (Elements of $W$)</th>
<th>$\hat{\theta}$</th>
<th>SE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum engineer’s estimates in millions</td>
<td>0.0023**</td>
<td>0.0011</td>
</tr>
<tr>
<td>Fraction overlapping time among projects in combination</td>
<td>0.0753***</td>
<td>0.0166</td>
</tr>
<tr>
<td>Fraction overlapping time $\times$ Sum engineer’s estimates in millions</td>
<td>-0.0035***</td>
<td>0.0009</td>
</tr>
<tr>
<td>Distance in KM among projects in combination</td>
<td>-0.0001***</td>
<td>0.0000</td>
</tr>
<tr>
<td>Same-type-auctions index</td>
<td>-0.0952***</td>
<td>0.0198</td>
</tr>
<tr>
<td>Regular bidder</td>
<td>0.0235</td>
<td>0.0122</td>
</tr>
<tr>
<td>Large bidder</td>
<td>0.0196</td>
<td>0.0178</td>
</tr>
<tr>
<td>Bidder type FE</td>
<td>YES</td>
<td>-</td>
</tr>
</tbody>
</table>

Units are scaled by sum of engendered estimates, positive $\kappa$ means lower cost (more cost synergy, larger complementarities) between projects.

though very small, these effects are all natural and highly consistent with our priors. While not reported in Table 6, we also include a vector of bidder type dummies in $\kappa(\cdot)$; signs on these vary, but suggest positive complementarities on aggregate as we quantify next.

To illustrate the economic significance of these parameter estimates, we next translate the parameter estimates $\hat{\theta}$ in Table 6 into estimates for the underlying complementarities $\kappa(\cdot)$ themselves. Specifically, we first construct, for each bidder $i$ in the sample, the estimated complementarity associated with the combination of projects for which $i$ bids. We then normalize this complementarity by the total size of projects in this combination, and analyze the distribution of these normalized complementarities across bidders.

Table 7 summarizes the results of this procedure, reporting quantiles of normalized complementarities for all bidders in our MDOT sample. As evident from Table 7, there is substantial heterogeneity in complementarities across bidders in the MDOT sample, with a joint win leading to cost savings of approximately 13 percent of combination size at the 5th (best) quantile of complementarities, transitioning to cost increases of approximately 6.6 percent at the 95th (worst) quantile. Recalling the parameter estimates in Table 6, we view these pattern as consistent with an underlying U-shaped cost curve, with completion costs falling until firm resources are fully employed and rising thereafter.

We conclude this section with a note on interpretation of Tables 6 and 7 under endogenous
Table 7: Distribution of complementarities across bidders

<table>
<thead>
<tr>
<th>Quantile rank</th>
<th>Quantile of normalized $\hat{\kappa}(Z_i, W_i)$:</th>
<th>Full sample</th>
<th>Two-auction subsample</th>
</tr>
</thead>
<tbody>
<tr>
<td>5th</td>
<td>-0.128731</td>
<td>-0.136240</td>
<td></td>
</tr>
<tr>
<td>10th</td>
<td>-0.108642</td>
<td>-0.102615</td>
<td></td>
</tr>
<tr>
<td>15th</td>
<td>-0.092941</td>
<td>-0.076672</td>
<td></td>
</tr>
<tr>
<td>20th</td>
<td>-0.080921</td>
<td>-0.061132</td>
<td></td>
</tr>
<tr>
<td>25th</td>
<td>-0.069927</td>
<td>-0.053015</td>
<td></td>
</tr>
<tr>
<td>30th</td>
<td>-0.061546</td>
<td>-0.051304</td>
<td></td>
</tr>
<tr>
<td>35th</td>
<td>-0.053289</td>
<td>-0.050354</td>
<td></td>
</tr>
<tr>
<td>40th</td>
<td>-0.050775</td>
<td>-0.049481</td>
<td></td>
</tr>
<tr>
<td>45th</td>
<td>-0.048571</td>
<td>-0.038151</td>
<td></td>
</tr>
<tr>
<td>50th</td>
<td>-0.039772</td>
<td>-0.033978</td>
<td></td>
</tr>
<tr>
<td>55th</td>
<td>-0.032824</td>
<td>-0.024801</td>
<td></td>
</tr>
<tr>
<td>60th</td>
<td>-0.023904</td>
<td>-0.017934</td>
<td></td>
</tr>
<tr>
<td>65th</td>
<td>-0.016951</td>
<td>-0.016113</td>
<td></td>
</tr>
<tr>
<td>70th</td>
<td>-0.007646</td>
<td>-0.002433</td>
<td></td>
</tr>
<tr>
<td>75th</td>
<td>0.005610</td>
<td>0.011742</td>
<td></td>
</tr>
<tr>
<td>80th</td>
<td>0.022205</td>
<td>0.037228</td>
<td></td>
</tr>
<tr>
<td>85th</td>
<td>0.037447</td>
<td>0.053367</td>
<td></td>
</tr>
<tr>
<td>90th</td>
<td>0.053890</td>
<td>0.056523</td>
<td></td>
</tr>
<tr>
<td>95th</td>
<td>0.066264</td>
<td>0.083075</td>
<td></td>
</tr>
</tbody>
</table>

“Normalized $\hat{\kappa}(Z_i, W_i; \hat{\theta})$” denotes estimated complementarity $\hat{\kappa}(Z_i, W_i; \hat{\theta})$ among projects bid by $i$ divided by the sum of engineer’s estimates among projects bid by $i$, with quantiles evaluated over the empirical distribution of $(Z_i, W_i)$ over all bidders and periods in the sample indicated. Negative fractions represent positive complementarities (lower costs). Thus the statement that the 50th quantile of normalized $\hat{\kappa}(Z_i, W_i)$ is $-0.039$ in the full sample means that for the median bidder a joint win would generate cost savings equal to approximately 3.9 percent of combination size, and similarly.
entry. In Appendix B, we embed the bidding model considered here within a fully specified
entry and bidding game, showing that our estimation strategy is robust to this extension.
Hence the parameter estimates reported in Table 6 remain valid even under entry. In inter-
preting Table 7, however, it is important to note that the distribution of complementarities
among projects in which bidders enter will differ from that which would arise if projects
were randomly assigned. In particular, insofar as bidders tend to bid for complementary
combinations, we would expect the distribution in Table 7 to be positively selected.

6 Counterfactuals

While the simultaneous first-price auction is clearly inefficient when bidders have combi-
natorial preferences, little is known about the magnitude of these inefficiencies in practice.
Furthermore, little is known either theoretically or empirically about the revenue properties
of the simultaneous first-price auction relative to other feasible multi-object mechanisms
such as the Vickrey-Clarke-Groves (VCG) auction.

As a first step toward answering these questions, we compare revenue and efficiency under
MDOT’s actual simultaneous low-bid first-price form with counterfactual outcomes which
would have arisen under a combinatorial VCG auction. We implement this comparison as
follows. First, for each bidder $i$ and letting $t$ in our estimation sample, we estimate $i$’s stan-
dalone cost vector by mapping $i$’s observed bid $b_{it}$ through the inverse bid function (7), taking
complementarity estimates obtained in Step 2 as given. \(^{28}\) We then simulate allocations, costs
of project completion, and payments to bidders under both the baseline simultaneous first-
price auction and the counterfactual combinatorial VCG auction, computing final completion
costs inclusive of cross-auction complementarities in both cases. \(^{29}\) Finally, we average these

\(^{28}\)In practice, a small number of simulated costs are either negative or implausibly large. To prevent bias
from these outliers, we replace standalone cost draws which are less than zero or more than 200 percent
of the engineer’s estimate with new bids from the bid distribution $\hat{G}_i(\cdot)$ estimated in Step 1, and their
corresponding costs.

\(^{29}\)In these simulations, we set MDOT’s effective reserve price for each project equal to 200 percent of the
MDOT engineer’s cost estimate; other plausible values generate very similar results.
payments and costs across auctions to obtain our aggregate counterfactual results.

Two patterns emerge from this exercise. First, as expected, the simultaneous first-price mechanism is socially inefficient, generating expected social costs of roughly $1.44 million per auction, versus $1.37 million per auction for the VCG mechanism. In level terms, this difference is nontrivial, translating to an average savings of roughly $65,000 per auction. Yet in percentage terms gains are relatively small: roughly 4.45 percent social cost savings relative to total project completion costs under the simultaneous first-price mechanism.

Second, and more strikingly, MDOT’s payments to bidders are substantially lower in the simultaneous first-price auction than they would be under VCG: $1.57 million per auction in the baseline simultaneous first-price auction, versus $1.78 million per auction under VCG. In other words, even though the VCG mechanism is more efficient socially, it would increase MDOT’s procurement costs by roughly 13.7 percent. On the whole, we view these findings as strong suggestive evidence that the simultaneous first-price mechanism in fact performs remarkably well in the MDOT marketplace.\textsuperscript{30}

\section{Conclusion}

Motivated by an institutional framework common in procurement applications, we develop and estimate a structural model of bidding in simultaneous first-price auctions, to our knowledge the first such in the literature. We analyze identification of this model, showing that excluded variation in either characteristics of rival bidders or characteristics of other auctions supports nonparametric identification of cross-object complementarities. Finally, we apply this framework to data on Michigan Department of Transportation highway construction and maintenance auctions. Our estimates suggest that winning a two-auction combination generates cost effects ranging from roughly 13 percent cost savings (relative to combination

\textsuperscript{30}This analysis is of course only partial in that we effectively hold entry behavior fixed across mechanisms. By construction, since the VCG auction leads to both lower social costs and higher MDOT payments, it must also generate greater profit to bidders. In equilibrium, this should translate into greater entry, which could at least somewhat mitigate the revenue losses we find above. In contrast, since new entrants are by definition marginal, we expect true efficiency gains to be similar to those reported above.
size) at the 5th percentile to roughly 6.6 percent cost increases at the 95th percentile, with combination costs increasing in joint size of, scheduling overlap between, and heterogeneity in work types in the combination. Despite these substantial complementarities, we find that switching to an efficient Vickrey-Clarke-Groves mechanism would generate only modest social gains: roughly four percent savings in social costs of project completion, while substantially increase MDOT’s expected procurement costs. We view this as strong suggestive evidence that simultaneous first-price auctions can perform relatively well even in environments with economically important complementarities. This observation may partially rationalize the widespread popularity of simultaneous first-price auctions in practice.

Appendix A: Proof of Proposition 1

The proof of Proposition 1 rests on two key claims. First, the first-order system (7) must be well-defined for almost every $b_i$ submitted by $i$, i.e. almost everywhere with respect to the measure induced by $G_i(\cdot|Z,W)$. Second, at almost every $b_i$ at which first order conditions hold, the first-order system (7) must be invertible. We establish each claim in turn.

First show that the first order system (7) is well-defined for almost every $b_i$ submitted by $i$. Recall that we can write bidder $i$’s objective as

$$\pi(v_i, b|K; Z,W) = (\Omega v_i + K - \Omega b)^TP_i(b|Z,W).$$

where $v_i$ and $K$ are given at the time of maximization. Note that the system (7) necessarily holds at any best response where $\pi(v_i, \cdot|K; Z,W)$ is differentiable and that Assumption 3 implies that each observed $b_i$ is a best response. Hence the system (7) will be well defined for almost every $b_i$ submitted by $i$ if and only if $\pi(v_i, \cdot|K; Z,W)$ is differentiable almost everywhere with respect to the measure on $B_i$ induced by $G_i(\cdot|Z,W)$. But under Assumption 4, $G_i(\cdot|Z,W)$ is absolutely continuous. To establish the claim, it thus suffices to show differentiability of $\pi(v_i, \cdot|K; Z,W)$ a.e. with respect to Lebesgue measure on $B_i$.

Clearly $(\Omega v_i + K - \Omega b)$ is differentiable in $b$. Thus differentiability of $\pi(v_i, \cdot|K; Z,W)$ at $b$ is equivalent to differentiability of $P_i(\cdot|Z,W)$ at $b$. Let $B_{-i}$ be the $L_i \times 1$ random vector describing maximum rival bids in the set of auctions in which $i$ participates. Again applying Assumption 4 to rule out ties, the probability $i$ wins combination $\omega$ at bid $b$ is

$$P^\omega(b|Z,W) = \Pr\{\cap_{\{m: \omega_m = 1 \}} 0 \leq b_{-i,m} \leq b_{i,m} \cap \{\cap_{\{m: \omega_m = 0 \}} b_{i,m} \leq B_{-i,m} < \infty \}).$$

For each $\omega \in \Omega_i$, let $b^\omega$ be the $(\sum \omega) \times 1$ sub-vector of $b$ describing $i$’s bids for objects in $\omega$, $B_{-i}^\omega$ be the $(\sum \omega) \times 1$ sub-vector of $B_{-i}$ describing maximum rival bids for objects in $\omega$, and $G_{-i}^\omega(b^\omega|Z,W)$ be the equilibrium joint c.d.f. of $B_{-i}^\omega$ at $(Z,W)$. Applying the formula for a rectangular probability
and simplifying, we can then represent $P_i(\cdot|Z,W)$ in the form
\[ P_{-i}^{\omega}(b|Z,W) = \sum_{\omega' \in \Omega} a_{\omega'}^{\omega} G_{-i}^{\omega}(b^{\omega'}|Z,W), \]
where each $a_{\omega'}^{\omega}$ is a known scalar (determined by $\omega$, $\omega'$) taking values in \{-1, 0, 1\}. But by absolute continuity each c.d.f. $G_{-i}^{\omega}(\cdot|Z,W)$ is differentiable a.e. (Lebesgue) in its support, and interpreted as a function from $B_i$ to $\mathbb{R}^{K_i}$, each $b^{\omega'}$ is continuously differentiable in $b$. Thus interpreted as a function from $B_i$ to $\mathbb{R}$, each $G_{-i}^{\omega}(b^{\omega'}|Z,W)$ is differentiable on a set of full Lebesgue measure in $B_{-i}$. The set of points in $B_i$ at which all $G_{-i}^{\omega}(b^{\omega'}|Z,W)$ are differentiable is the intersection of points in $B_i$ at which each $G_{-i}^{\omega}(b^{\omega'}|Z,W)$ is differentiable, i.e. the intersection of a finite collection of sets of full Lebesgue measure in $B_i$. But from above differentiability of $G_{-i}^{\omega}(b^{\omega'}|Z,W)$ for all $\omega'$ implies differentiability of $P_{-i}^{\omega}(b|Z,W)$. Hence $P_{-i}^{\omega}(\cdot|Z,W)$ is differentiable on a set of full Lebesgue measure in $B_i$. This in turn implies differentiability of $\pi(v, \cdot|K;Z,W)$ a.e. with respect to the measure on $B_i$ induced by $G_{i}(\cdot|Z,W)$, as was to be shown.

We next establish that the first-order system (7) must yield a unique solution $\tilde{v}$ for almost every $b_i$ submitted by $i$. Let $\tilde{B}_i$ be the set of points in $B_i$ at which $\pi(\cdot, \cdot|K;Z,W)$ is differentiable in $b_i$; from above, $\tilde{B}_i$ is a subset of full Lebesgue measure in $B_i$. Choosing any $b \in \tilde{B}_i$ and rearranging (7) yields
\[ \nabla_b \Gamma_{i}(b|Z,W) \tilde{v} = \nabla_b \Gamma_{i}(b|Z,W)b + \Gamma_{i}(b|Z,W) - \nabla_b P_i(b|W,Z)^T K_i. \]

Hence uniqueness of $\tilde{v}$ is equivalent to invertibility of the $L_i \times L_i$ matrix $\nabla_b \Gamma_{i}(b|Z,W)$. Recall that $\Gamma_{i}(b|Z,W)$ is an $L_i \times 1$ vector whose $l$th element describes the probability that bid vector $b$ wins auction $l$. Note that $b \in \tilde{B}_i$ rules out ties at $b$. Thus for $b \in \tilde{B}_i$ the $m$th element of $\Gamma_{i}(b|Z,W)$ is the marginal c.d.f. $G_{-i,m}(b|Z,W)$ of $B_{-i,m}$, from which it follows that $\nabla_b \Gamma_{i}(b|Z)$ is a diagonal matrix whose $m$, $m$th element is the marginal p.d.f. $g_{-i,m}(b|Z,W)$ of $B_{-i,m}$. Hence $\nabla_b \Gamma_{i}(b|Z,W)$ will be invertible at $b$ if and only if $g_{-i,m}(b|Z,W) > 0$ for all $m = 1, \ldots, L_i$.

But by hypothesis each submitted bid $b_i$ is a best response to rival play at $(Z,W)$ for some $(v,K)$. Suppose that there exists an $\epsilon > 0$ such that $g_{-i,m}(\cdot|Z,W) = 0$ on $(b_{im} - \epsilon, b_i]$. Then player $i$ could infinitesimally reduce $b_{im}$ without affecting either $\Gamma_{i}$ or $P_i$, a profitable deviation for any $(v,K)$. Hence we must have $g_{-i,m}(\cdot|Z,W) > 0$ almost everywhere (Lebesgue) in the support of $B_i$. By Assumption 4, this in turn implies $g_{-i,m}(\cdot|Z,W) > 0$ for almost every $b_i$ submitted by $i$. Since $m$ was arbitrary, we must have $\nabla_b \Gamma_{i}(b_i|Z,W)$ invertible for almost every bid $b_i$ submitted by $i$. Hence for almost every $b_i$ submitted by $i$ there will exist a unique $\tilde{v}$ satisfying (7) at $b_i$, given by
\[ \tilde{v} = b_i + \nabla_b \Gamma_{i}(b_i|Z,W)^{-1} \Gamma_{i}(b_i|Z,W) + \nabla_b \Gamma_{i}(b_i|Z,W)^{-1} \nabla_b P_i(b_i|W,Z)^T K_i. \]

The RHS of this expression is identified up to $K$, establishing the claim.

**Appendix B: Entry**

In this Appendix, we formally embed the bidding model we describe above within a two-stage entry-plus-bidding model parrelling those considered by Levin and Smith (1994), Krasnokutskaya and Seim (2011), Athey et al. (2011), Moreno and Wooders (2011), Li and Zhang (2015), and Groeger (2014) among others. This discovery process proceeds as follows.
At the beginning of the game, each bidder $i$ is endowed with a $L \times 1$ standalone valuation vector $V_i^0$ drawn by nature from a joint distribution $F_i^0$. However, realizations of $V_i^0$ are \textit{ex ante unknown} to $i$ and can be discovered by $i$ only through costly entry.

Specifically, let $S$ denote the power set of $L$, i.e. the set of all sets of auctions in which bidder $i$ could enter. Let $S \subseteq \mathcal{S}$ denote a particular subset of auctions $S \in \mathcal{L}$. Suppose that, at the beginning of Stage 1, each bidder $i$ observes a $2^L \times 1$ vector of private combinatorial entry costs $C_i$, with element $C_i^S$ of $C_i$ describing the total cost $i$ must incur to enter the set of auctions $S \in \mathcal{S}$. This cost vector $C_i$ satisfies the following properties:

\textbf{Assumption 7 (Private Entry Costs). For each bidder $i$, $C_i$ is drawn independently of $V_i^0$ from cost distribution $F_{C,i}$ with support on a compact, convex set $C_i \subset \mathbb{R}^{2^L}$, with $C_i$ private information, $F_{C,i}$ common knowledge, and cost draws independent across bidders: $C_i \perp C_j$ for all $i, j$.}

Having observed $C_i$, bidder $i$ chooses a set of auctions $\mathcal{L}_i \in \mathcal{S}$ in which to enter, pays the corresponding entry cost $C_i^{\mathcal{L}_i}$, and proceeds to Stage 2. Then at the beginning of Stage 2, Bidder $i$ observes the realizations of her standalone valuation vector $V_i \equiv (V_i^0)_{l \in \mathcal{L}_i}$ for the auctions $l \in \mathcal{L}_i$ in which she has entered. Lastly, bidder $i$ submits a single bid $b_{il}$ for each object $l$ in her entry set $\mathcal{L}_i$. The bidding subgame then proceeds exactly as described in the main text.

While the combinatorial nature of the entry problem renders notation somewhat involved, this model is in fact the natural combinatorial generalization of the canonical entry and bidding models cited above. Specifically, when valuations and entry costs are additively separable and the entry cost distributions $F_{C,i}$ are atomistic, we obtain the mixed-strategy entry models considered by Levin and Smith (1994), Krasnokutskaya and Seim (2011), and Athey et al. (2011). Alternatively, if auctions are separable and entry costs are continuously distributed, we obtain a pure-strategy value discovery model parallel to Moreno and Wooders (2011), Li and Zhang (2015), and Groeger (2014). The structure we consider here generalizes these canonical value-discovery models to our substantially richer setting involving both common-knowledge complementarities and (potentially) combinatorial entry costs.

We now turn to consider equilibrium entry behavior. Following Milgrom and Weber (1985), define a \textit{distributional entry strategy} for player $i$ as a measure $\sigma_i^E$ over $C_i \times S$ whose marginal over $C_i$ is $F_{C,i}$, with $\sigma^E = (\sigma_1^E, \ldots, \sigma_N^E)$ a profile of distributional entry strategies. Then any Bayes-Nash equilibrium must have the following form.

Suppressing generic object, bidder, and combination characteristics, let a market structure $M = (S_1, \ldots, S_N)$ now denote any vector of participation sets for each bidder. Each market structure $M$ gives rise to a different simultaneous bidding subgame, the structure of which parallels that described in the main text. Let $\sigma^M = (\sigma_1^M, \ldots, \sigma_N^M)$ be any profile of Bayesian Nash equilibrium bidding strategies in the simultaneous bidding subgame arising under market structure $M$; as in the main text, we focus on pure strategies for simplicity, although this is inessential.

Recall that bidder $i$’s standalone valuations $V_i$ are unknown at the time of entry. Hence, taking subgame strategies $\sigma^M$ as given, the \textit{ex ante} (pre-entry) expected profit bidder $i$ associates with market structure realization $M$ is given by

$$
\Pi_i(M) = \int_{V_i} \left[ (V_i - \sigma_i^M(V_i))^T \Gamma_i(\sigma_i(V_i)|M) + \kappa_i^T P_i(\sigma_i^M(V_i)|M) \right] F_i(dV_i|M), \tag{25}
$$

where, as in the main text, the vectors $\Gamma_i(b_i|M)$ and $P_i(b_i|M)$ denote the expected marginal and combinatorial win probabilities $i$ associates with bid vector $b_i$, taking rival bidding strategies $\sigma^M_{-i}$ as given.
Now consider the entry decision by bidder $i$. With slight abuse of notation, let $S_{-i} = (S_j)_{j \neq i}$ denote any realizations of participation sets for $i$’s rivals, so that we may write market structure as $M = (S_i, S_{-i})$. Taking rival entry strategies $\sigma^E$ as given, we may write $i$’s Stage 1 expected payoff from entering auction combination $S_i \in S$ as

$$\Xi(S, \sigma^E_{-i}) = E[\Pi_i(S, S_{-i})|\sigma^E_{-i}],$$

where the expectation is over ex ante unknown rival entry decisions $S_{-i}$. Hence, in Stage 1, bidder $i$ will optimally choose the participation set $\mathcal{L}_i$ maximizing her expected payoff net of entry costs:

$$\mathcal{L}_i = \arg\max_{S \in S} \Xi(S, \sigma^E_{-i}) - C^S_i. \quad (26)$$

The Stage 1 action set for each bidder is the finite set $S$, and private entry cost vectors $C_i, C_j$ are independent across bidders $i, j$. Hence, taking the continuation payoff functions $\Pi_1(M), ..., \Pi_N(M)$ as given, by Proposition 1 of Milgrom and Weber (1985) there exists an equilibrium in distributional strategies for the Stage 1 entry game. So long as continuation payoffs $\Pi_1(M), ..., \Pi_N(M)$ are themselves generated from play of a Bayes-Nash equilibrium under every market structure $M$, this in turn will constitute an equilibrium of the overall entry and bidding game.

In general, equilibrium Stage 1 entry may be in either pure or mixed strategies. If, however, we add the restriction that $F_{C_i}$ is atomless on $\mathcal{C}_i$ for each $i$, then Proposition 4 of Milgrom and Weber (1985) implies existence of a equilibrium in which bidders play pure entry strategies.

How does this entry and bidding subgame shape our understanding of the bid-stage subgame we consider in the main text? Clearly, when entry is endogenous, each bidder $i$ will tend to enter in combinations for which she expects either high standalone valuations, or positive complementarities, or both. Furthermore, as noted above, entry may be in either pure or mixed strategies, and there may exist many entry equilibria. So long as, however, only one set of equilibrium bidding strategies $\sigma^M$ is played conditional on realization of each potential market structure $M$, we may proceed with identification maintaining Assumptions 1-5 as in the main text.

Furthermore, since $C_j \perp V_j$ for all bidders $j$, the distribution of $i$’s post-entry private information is invariant to the realizations $S_{-i}$ of entry decisions by $i$’s competitors, variation in $S_{-i}$ will be will be effectively exogenous and thus excludable with respect to $F_i$. In other words, so long as there exists at least some variation in rival entry decisions conditional $(X, Z_i, W_i)$—whether induced by variation in rival entry costs, mixed entry strategies as in Athey et al. (2011) and Levin and Smith (1994), or mixing across entry equilibria—then equilibrium entry will induce precisely the form of variation required for our identification argument. In this sense, the entry model described above provides a formal equilibrium justification for the key exclusion restriction (Assumption 5) on which we base our nonparametric identification argument.

Finally, note that the equilibrium entry conditions (26) in principle provide an additional set of identifying restrictions on complementarities. Specifically, we know that the combination which bidder $i$ actually entered must yield the highest ex ante profit among all combinations which $i$ could have entered. Furthermore, under the hypothesis $K_i = \kappa_i(Z_i, W_i)$, there exists a unique candidate $\hat{\Pi}_i(M, K_i)$ for the ex ante profit function $\Pi_i(M)$, which is identified up to $K_i$. Exploiting necessary conditions for optimality—for instance, that bidder $i$ should not gain in expectation by adding or removing one auction from her participation set—one could translate (26) into a set of restrictions on complementarities $K_i$ and entry costs $C_i$ jointly. In principle, these would provide further identifying information on $K_i$. Unfortunately, these conditions also involve numerous high-dimensional integrals, evaluation of which would involve significant computational costs. In practice, we therefore focus on restrictions on $K_i$ induced by the bidding model, without reference
Appendix C: Unobserved auction heterogeneity

In this Appendix, we discuss how our identification results can be extended to allow for additively separable unobserved auction heterogeneity. Specifically, suppose that bidder $i$’s standalone valuation $V_{il}$ in auction $l$ is

$$V_{il} = U_{il} + A_l,$$

where $A_l$ is the unobserved heterogeneity in auction $l$, common knowledge to bidders, and $U_{il}$ is private information for each bidder $i$. As is well known in the literature, if $U_{il}$ are independent across $i$, independent of $A_l$, and each $A_l$ has a log-concave density, then the vector of standalone valuations $(V_{1l}, \ldots, V_{Nl})$ is affiliated.

We further assume that the distribution of unobserved heterogeneity $A_l$ in auction $l$ depends only on the characteristics $X_l$ of object $l$:

$$A_l \perp Z \mid X, W, \quad A_l \perp W \mid X, U_{il} \perp A_l \mid X, Z, W, \quad (27)$$

$$A_l \perp X_l \mid X, W, \quad A_l \perp A_{\tilde{m}} \mid X, W \quad \text{for } l \neq \tilde{m}, \quad (28)$$

Denote $A = (A_1, \ldots, A_L)^\top$. Under regularity conditions analogous to those in our main text, bidding at observables $X, Z, W$ and unobservable realization $A$ must satisfy the F.O.C.

$$U_i + A = B_i + \nabla_b \Gamma_i(B_i|X, Z, W, A)^{-1} \times \Gamma_i(B_i|X, Z, W, A)$$

$$- \nabla_b \Gamma_i(B_i|X, Z, W, A)^{-1} \times \nabla_b P_i(B_i|X, Z, W, A)^T K_i. \quad (29)$$

Also note that, if the strategy profile $\sigma^h(\cdot|X, Z, W)$ is a Bayes-Nash equilibrium at observable market characteristics $(X, Z, W)$ and unobservables $A = 0$, then the strategy profile

$$\sigma(\cdot|X, Z, W, A) \equiv \sigma^h(\cdot|X, Z, W) + A$$

is a Bayes-Nash equilibrium at observables $(X, Z, W)$ and unobserved heterogeneity realization $A$. To see why this is so, note that conditional on $(X, Z, W)$ and $A$, the strategy profiles $\sigma(\cdot|X, Z, W, A)$ yields the same allocations and payoffs as does the strategy profile $\sigma^h(\cdot|X, Z, W)$. We assume throughout that, conditional on $(X, Z, W)$, the same “fundamental” strategies $\sigma^h(\cdot|X, Z, W)$ are played for all realizations of $A$, so that variation in $A$ shifts bids in each auction additively.

Under these assumptions, we can use techniques in Krasnokutskaya (2011) to establish that under suitable normalizations of mean bids within each auction $l$, both the distribution of $B_{il}$ conditional on $A_l, X, Z, W$ and the distribution of $A_l$ conditional on $X, W$ are identified from the distribution of $B_{il}$ conditional on $X, Z, W$ (recall that, conditional on $X, W$, $A_l$ is independent of $Z$). In turn, by independence of $A_l$ across auctions conditional on $W, X$, identification of each marginal $A_l|X, W$ implies identification of the distribution of the whole vector $A_l|X, W$.

The next step is to identify the joint distribution of the bid vector $B_i$ conditional on $X, Z, W, A$ from the following information known to the econometrician: (i) the joint distribution of $B_i$ conditional on $Z, X, W$; (ii) the distribution of $A|X, W$. Here we employ the fact that

$$B_i = B^h_i + A.$$

Thus, given identification of distributions of $B^h_i|X, Z, W$ and $A|X, Z, W$, the distribution of $B^h_i|X, Z, W$
is identified by a standard deconvolution argument. Note that the distribution of \( B_i^h | X, Z, W \) does not depend on \( A \) and, thus, the distribution of \( B_i | X, Z, W, A \) is simply a location shift (by \( A \)) of the distribution of now identified \( B_i^h | X, Z, W \).

The rest of the identification strategy is analogous to our baseline case. Namely, we can apply either the identification strategy based on Assumption 5 (varying rival characteristics \( Z_{-i} \)), or the identification strategy based on Assumption 6 (varying characteristics of other auctions \( X_{-i} \)) to identify distributions of “undisturbed” standalone valuations \( U_{il} \) and complementarities \( K_i \).

Together with the conditions above on unobserved auction heterogeneity, the former approach would give us that \( E[U_i + A | X, Z, W] \) does not depend on \( Z_{-i} \), while the latter would give us that \( E[U_{il} + A_i | X, Z, W] \) does not depend on \( X_{-i} \). In either case, the system for determining \( K_i(i, W_i) \) will be exactly the same as described in Section 3.

**Appendix D: Complementarities depending on \( V \)**

In this appendix, we explore prospects for generalizing our non-parametric identification results to the case where complementarities are additively separable functions of standalone valuations. In other words, conditional on \( Z, W, X \) the complementarities are stochastic but their randomness can be fully explained by the standalone valuations. As a special case, we consider a scenario when these functions are affine in standalone valuations. Such a case could arise if, for instance, winning two auctions together increases \( i \)'s valuation for one or both objects by a fixed percentage.

**Notation and definitions** We say complementarities are **additively separable in standalone valuations** if for each \( \omega \) that contains at least two non-zero components (that is, \( \| \omega \|^2 \geq 2 \)), the complementarity for outcome \( \omega \) is a function of the vector of standalone valuations \( v_i = (v_{i1}, v_{i2}, \ldots, v_{iL_i})^T \) such that

\[
K^\omega(v_i, Z_i, W_i) = \sum_{l : \omega_l = 1} \phi_l(v_{il}, Z_i, W_i) + \bar{K}^\omega(Z_i, W_i) \tag{30}
\]

for some functions \( \phi_l, l = 1, \ldots, L \). If each function \( \phi_l \) is linear in \( v_{il} \), then we obtain the special case of **complementarities affine in \( v_i \)**:

\[
K^\omega(v_i, Z_i, W_i) = \sum_{l : \omega_l = 1} \delta^l(Z_i, W_i)v_{il} + \bar{K}^\omega(Z_i, W_i), \quad \text{if } \|\omega\|^2 \geq 2. \tag{31}
\]

As usual, if \( \omega \) contains at most one component equal to one (that is, \( \| \omega \|^2 \leq 1 \)), then we set \( K^\omega(v_i, Z_i, W_i) \equiv 0 \). An interesting special case of (31) is when all \( \delta^l \) are identical and \( \bar{K}^\omega = 0 \) for any \( \omega \). This case describes the situation of a constant relative complementarity – that is, when \( K^\omega(v_i, Z_i, W_i) \) is a constant ratio of the additive valuation.

Now assume that complementarities are affine in \( v_i \), and define an \( L_i \times 1 \) vector \( \delta(Z_i, W_i) \) and an \( L_i \times L_i \) matrix \( D(\delta(Z_i, W_i)) \) as follows:

\[
\delta(Z_i, W_i) = (\delta^1(Z_i, W_i), \delta^2(Z_i, W_i), \ldots, \delta^{L_i}(Z_i, W_i))^T \quad \text{and} \quad D(\delta(Z_i, W_i)) = \text{diag}(\delta^1(Z_i, W_i), \delta^2(Z_i, W_i), \ldots, \delta^{L_i}(Z_i, W_i)).
\]

To write this in a convenient vector-matrix notation, let \( A_i \) denote the \( 2^{L_i} \times 2^{L_i} \) matrix such that its submatrix \( (a_{ij})_{l,j=L_i+2, \ldots, 2M} \) coincides with the identity matrix of size \( 2^{L_i} - L_i - 1 \), with all the
other elements of $A_i$ being 0. We then have

$$K(v_i, Z_i, W_i) = A_i \Omega_i D(\delta(Z_i, W_i)) v_i + \bar{K}(Z_i, W_i),$$

where $\bar{K}(Z_i, W_i)$ denotes the $2^{L_i} \times 1$ vector of constant components in the complementarities (obviously, $\bar{K}(Z_i, W_i) \in K_i$). Clearly, the rank of matrix $A_i \Omega_i$ is equal to $L_i$.

As can be seen, the functional form of complementarities does not depend on $Z_{-i}$. As we show below, under weak conditions there is enough variation in $Z_{-i} | Z_i, W, X$ to determine the linear (in $v_{il}$) part of complementarities as well as the constant part.

**Non-parametric identification** Using the first-order conditions and taking into account the form of $K(v_i, Z_i, W_i)$, obtain

$$v_i = b_i + [\nabla_b \Gamma_i(b_i | Z_{-i})]^{-1} \Gamma_i(b_i | Z_{-i}) - [\nabla_b \Gamma_i(b_i | Z_{-i})]^{-1} \nabla_b P_i(b_i | Z_{-i})^T A_i \Omega_i D(\delta) v_i + \bar{K},$$

where for notational simplicity conditioning on $Z_i$, $W, X$ is omitted from the notation in the rest of this Appendix. Rewrite that system of equations by collecting all terms with $v_i$ on the left-hand side:

$$\left( I_{L_i} + [\nabla_b \Gamma_i(b_i | Z_{-i})]^{-1} \nabla_b P_i(b_i | Z_{-i})^T A_i \Omega_i D(\delta) \right) v_i = b_i + [\nabla_b \Gamma_i(b_i | Z_{-i})]^{-1} \Gamma_i(b_i | Z_{-i}) \bigg| \bigg| \nabla_b P_i(b_i | Z_{-i})^T A_i \Omega_i D(\delta),$$

and introduce a notation for the matrix in front of $v_i$ on the left-hand side:

$$\Pi(b_i, \delta, Z_{-i}) \equiv I_{L_i} + [\nabla_b \Gamma_i(b_i | Z_{-i})]^{-1} \nabla_b P_i(b_i | Z_{-i})^T A_i \Omega_i D(\delta).$$

Define $\Delta(Z_{-i})$ as the set of $\delta \in \mathbb{R}^{L_i}$ such that

$$\Pi(b_i, \delta, Z_{-i})$$

is non-singular for almost all $b_i$.

This set is non-empty as e.g. $0 \in \Delta(Z_{-i})$. If $\delta \in \Delta(Z_{-i})$, then we can multiply the system from the left by $\Pi(b_i, \delta, Z_{-i})^{-1}$ resulting in

$$v_i = \Pi(b_i, \delta, Z_{-i})^{-1} b_i + \Pi(b_i, \delta, Z_{-i})^{-1} \bigg| \bigg| \nabla_b \Gamma_i(b_i | Z_{-i})^{-1} \Gamma_i(b_i | Z_{-i}) \bigg| \bigg| \nabla_b P_i(b_i | Z_{-i})^T A_i \Omega_i D(\delta),$$

Assuming that $\delta \in \Delta(Z_{-i})$ and carrying on with fixed $Z_i, W, X$, let us denote

$$D_1(\delta, Z_{-i}) \equiv E_{B_i} \left[ \Pi(B_i, \delta, Z_{-i})^{-1} B_i | Z_{-i} \right] + E_{B_i} \left[ \Pi(B_i, \delta, Z_{-i})^{-1} \nabla_b \Gamma_i(B_i | Z_{-i})^{-1} \Gamma_i(B_i | Z_{-i}) | Z_{-i} \right].$$

$$D_2(\delta, Z_{-i}) \equiv E_{B_i} \left[ \Pi(B_i, \delta, Z_{-i})^{-1} \nabla_b \Gamma_i(B_i | Z_{-i})^{-1} \nabla_b P_i(B_i | Z_{-i})^T | Z_{-i} \right].$$

Keeping $Z_i, W, X$ fixed, let us draw another value $Z'_{-i}$ from $Z_{-i} | Z_i, W, X$. Due to the assumptions made on the distribution of the standalone valuations, $E[V_i | Z_i, Z_{-i}, W, X] = E[V_i | Z_i, Z'_{-i}, W, X]$. Therefore, for $\delta \in \Delta(Z_{-i}) \cap \Delta(Z'_{-i}),$

$$D_1(\delta, Z'_{-i}) - D_1(\delta, Z_{-i}) = (D_2(\delta, Z'_{-i}) - D_2(\delta, Z_{-i})) \bar{K}.$$

For fixed $Z_i, W, X$, this system has $2^{L_i} - 1$ unknowns ($L_i$ in $\delta$ and $2^{L_i} - L_i - 1$ in $\bar{K}$) and $L_i$
equations. This gives us the following result.

**Proposition 3.** Suppose that for \((Z_i, W, X) \in Z_i \times W \times X\), there exist \(J+1 \geq (2^{L_i} - 1)/L_i + 1\) vectors \(Z_{-i,0}, Z_{-i,1}, \ldots, Z_{-i,J}\) in the support \(Z_{-i}|Z_i, W, X\) such that there is a unique \(\delta \in \bigcap_{j=0}^{J} \Delta(Z_{-i,j})\) and a unique \(\kappa \in \mathcal{K}_i\) that solve the system of \(J \cdot L_i\) equations

\[
D_1(\delta, Z_{-i,j}) - D_1(\delta, Z_{-i,0}) = (D_2(\delta, Z_{-i,j}) - D_2(\delta, Z_{-i,0})) \kappa, \quad j = 1, \ldots, J.
\]

Then the values of \(\delta(Z_i, W_i)\) and \(\tilde{K}(Z_i, W_i)\) are identified, and thus, the complementarity function is identified for these values of \(Z_i, W_i\).

System (32) is non-linear in \(\delta\). However, for each fixed \(\delta \in \bigcap_{j=0}^{J} \Delta(Z_{-i,j})\), this system is linear in \(\kappa\). Proposition 3 implies that in the case of identification it is not possible to have a situation when for different \(\delta_1\) and \(\delta_2\), \(\delta_1, \delta_2 \in \bigcap_{j=0}^{J} \Delta(Z_{-i,j})\), system (32) has solutions \(\kappa_1 \in \mathcal{K}_i\) and \(\kappa_2 \in \mathcal{K}_i\), respectively. Thus, in this sense the question of identification of \(\delta(Z_i, W_i)\) and \(\tilde{K}(Z_i, W_i)\) comes down to the question of the existence of a solution to a system of linear equations; (32) can have a solution \(\kappa\) for one \(\delta\) only, and for that \(\delta\) it has to be unique. Using the Kronecker-Capelli theorem, which gives the necessary and sufficient conditions for the existence of a solution to a system of linear equations, and also the necessary and sufficient conditions for the uniqueness of such a solution, we formulate the identification result in the Proposition 4 below.

Before we proceed to Proposition 4, let us rewrite (32) in a more convenient way. At the moment \(\mathcal{K}_i\) has to satisfy certain restrictions (namely, the first \(L_i + 1\) components of this vector are 0) and we first want to rewrite it through an unrestricted parameter to apply certain tools from algebra. Let \(E_i\) denote the \(2^{L_i} \times (2^{L_i} - L_i - 1)\) matrix such that its submatrix \((\bar{e}_{ij})_{i=0}^{2^{L_i}}, \bar{e}_{ij} \in \{1, \ldots, 2^{L_i} - L_i - 1\}\) coincides with the identity matrix of size \(2^{L_i} - L_i - 1\), and all its other elements (that is, all the elements in the first \(L_i + 1\) rows) are equal to zero. For every \(\kappa \in \mathcal{K}_i\) there is a unique \(\bar{\kappa} \in \mathbb{R}^{2^{L_i} - L_i - 1}\) such that

\[
\kappa = E_i \bar{\kappa}.
\]

Obviously, this \(\bar{\kappa}\) is a parameter that does not have to satisfy any prior restrictions. It is formed by the last \(2^{L_i} - L_i - 1\) values in \(\kappa\). System (32) can equivalently be written as

\[
D_1(\delta, Z_{-i,j}) - D_1(\delta, Z_{-i,0}) = (D_2(\delta, Z_{-i,j})E_i - D_2(\delta, Z_{-i,0})E_i) \bar{\kappa}, \quad j = 1, \ldots, J,
\]

with \(\bar{\kappa} \in \mathbb{R}^{2^{L_i} - L_i - 1}\). For a fixed \(\delta\), system (32) is linear in \(\kappa\), has the \(J \cdot L_i \times 2^{L_i}\) matrix of coefficients, and imposes restrictions on the solution \(\kappa\) by requiring that \(\kappa \in \mathcal{K}_i\). Its equivalent representation (33) is linear in \(\bar{\kappa}\) for a fixed \(\delta\), has the \(J \cdot L_i \times (2^{L_i} - L_i - 1)\) matrix of coefficients, and does not impose any restrictions on the solution \(\bar{\kappa}\) \(\in \mathbb{R}^{2^{L_i} - L_i - 1}\). This allows us to apply the Kronecker-Capelli theorem to system (33) in a straightforward way.

**Proposition 4.** Suppose that for \((Z_i, W_i, X) \in Z_i \times W_i \times X\), there exist \(J+1 \geq (2^{L_i} - 1)/L_i + 1\) vectors \(Z_{-i,0}, Z_{-i,1}, \ldots, Z_{-i,J}\) in the support \(Z_{-i}|Z_i, W, X\) such that there is a unique \(\delta \in \bigcap_{j=0}^{J} \Delta(Z_{-i,j})\) that satisfies the following two conditions:

1. First,

\[
\text{rank } (|M_1(\delta) | M_2(\delta)) = \text{rank } (M_2(\delta)),
\]

52
where $M_2(\delta)$ denotes the $J \cdot L_i \times (2^{L_i} - L_i - 1)$ matrix
$$
M_2(\delta) \equiv \begin{bmatrix}
D_2(\delta, Z_{-i,1}) E_i - D_2(\delta, Z_{-i,0}) E_i \\
\vdots \\
D_2(\delta, Z_{-i,J}) E_i - D_2(\delta, Z_{-i,0}) E_i
\end{bmatrix},
$$
and $M_1(\delta)$ denotes the $J \cdot L_i \times 1$ vector
$$
M_1(\delta) \equiv \begin{bmatrix}
D_1(\delta, Z_{-i,1}) - D_1(\delta, Z_{-i,0}) \\
\vdots \\
D_1(\delta, Z_{-i,J}) - D_1(\delta, Z_{-i,0})
\end{bmatrix}.
$$

2. Moreover, this $\delta$ is such that $M_2(\delta)$ has full column rank:
$$
\text{rank}(M_2(\delta)) = 2^{L_i} - L_i - 1.
$$

Then the values of $\delta(Z_i, W_i)$ and $\bar{K}(Z_i, W_i)$ are identified, and thus, the complementarity function is identified for these values of $Z_i, W_i$.

Condition (34) requires that in system (33), the rank of the matrix of coefficients $M_2(\delta)$ is equal to the rank of the augmented matrix $[M_1(\delta) | M_2(\delta)]$ for one $\delta$ only. The Kronecker-Capelli theorem guarantees then that (33) has a solution $\bar{\kappa}$ for that $\delta$ only. Condition (35) then guarantees this $\bar{\kappa}$ is determined uniquely, and, thus, $\kappa = E_i \bar{\kappa}$ is determined uniquely.

Note that all the identification conditions in Proposition 4 are formulated in terms of $\delta$. The closed form for $\delta(Z_i, W_i)$ cannot be found, but in practice one can find $\delta(Z_i, W_i)$ and $\bar{K}(Z_i, W_i)$ by solving, e.g., the following optimization problem:
$$
\min_{\delta \in \bigcap_{j=0}^{J} \Delta(Z_{-i,j}), \bar{\kappa} \in \mathbb{R}^{2^{L_i} - L_i - 1}} Q(\delta, \bar{\kappa}, Z_i, W, X),
$$
where
$$
Q(\delta, \bar{\kappa}, Z_i, W, X) \equiv (M_1(\delta) - M_2(\delta) \bar{\kappa})^T (M_1(\delta) - M_2(\delta) \bar{\kappa}).
$$

Appendix E: Partial identification relaxing conditions on equilibrium bidding in Assumption 4

Our point identification result for the vector-function of complementarities $\kappa_i(Z_i, W, X)$ and the conditional distribution of $V_i|Z_i, W, X$ relied on the first order conditions obtained from bidder’s optimization of the payoff function. To derive those equations we employed the absolute continuity of the bid distribution functions $G_i$. That, in particular, eliminated the possibility of bidders playing atoms in the equilibrium. In this appendix, we want to illustrate an approach to the identification question without either the absolute continuity or common support assumptions on $G_i$ maintained in Assumption 4. Our identification method is based on using inequalities for bidder’s best responses and employing the exclusion restrictions in Assumption 5 to obtain bounds on $\kappa_i(Z_i, W_i)$ and the marginal CDFs of $V_{il}|Z_i, X$.

Let us fix $(Z_i, W_i, X) \in Z_i \times W \times X$. For each realization $(Z_{-i}, W_{-i})$ in the support of
thus, a change in the location of the "ties", i.e., in the 
where we used the assumption that the ties are broken independently across auctions at 

\[ b \] is (weakly) increasing in \( i \)

for \( \epsilon > 0 \) and obtain from (37) that

\[ v_{il} (\Gamma_{i,l}(b_{il}|Z_{-i}) - \Gamma_{i,l}(b_{il} + \epsilon|Z_{-i})) + (P_i(b_i|Z_{-i}) - P_i(b_i + \epsilon|Z_{-i}))^T \kappa(Z_i, W_i) \geq \]

\[ b_{il} \Gamma_i(b_{il}|Z_{-i}) - (b_{il} + \epsilon) \Gamma_i(b_{il} + \epsilon|Z_{-i}) \quad \forall b \in B_i. \] (37)

\[ v_i^T \Gamma_i(b_i|Z_{-i}) - b_i^T \Gamma_i(b_i|Z_{-i}) + P_i(b_i|Z_{-i})^T \kappa(Z_i, W_i) \geq \]

\[ v_i^T \Gamma_i(b_i|Z_{-i}) - b_i^T \Gamma_i(b_i|Z_{-i}) + P_i(b_i|Z_{-i})^T \kappa(Z_i, W_i) \quad \forall b \in B_i. \] (36)

We can rewrite the best-response requirement (36) equivalently as follows:

\[ v_i^T (\Gamma_i(b_i|Z_{-i}) - \Gamma_i(b|Z_{-i})) + (P_i(b_i|Z_{-i}) - P_i(b_i|Z_{-i}))^T \kappa(Z_i, W_i) \geq \]

\[ b_i^T \Gamma_i(b_i|Z_{-i}) - b_i^T \Gamma_i(b_i|Z_{-i}) \quad \forall b \in B_i. \] (37)

The set of \((v_i, \kappa(Z_i, W_i))\) that satisfy the linear inequalities (37) is clearly convex. If the bidding set \( B_i \) is a continuum, then (37) represents a continuum of linear inequalities in \((V_i, \kappa(Z_i, W_i))\). If the convex set described by (37) is a singleton, then we are in the situation of point identification. Otherwise, we are in a scenario of partial identification. This convex set is fairly difficult to describe in a closed form. A much easier task is to describe its superset and then use it to derive bounds on the c.d.f.s of \( V_i|Z_i, X \).

We start by obtaining a closed form for a superset of the identified set for \( \kappa(Z_i, W_i) \). To construct this superset, we consider in (37) only those \( b \) that are different from the observed equilibrium vector \( b_i \) in one component. Namely, we first consider

\[ b = b_i + \epsilon e_l \in B_i \]

for \( \epsilon > 0 \) and obtain from (37) that

\[ v_{il} (\Gamma_{i,l}(b_{il}|Z_{-i}) - \Gamma_{i,l}(b_{il} + \epsilon|Z_{-i})) + (P_i(b_i|Z_{-i}) - P_i(b_i + \epsilon|Z_{-i}))^T \kappa(Z_i, W_i) \geq \]

\[ b_{il} \Gamma_i(b_{il}|Z_{-i}) - (b_{il} + \epsilon) \Gamma_i(b_{il} + \epsilon|Z_{-i}) \]

where we used the assumption that the ties are broken independently across auctions at \( b_i \), and thus, a change in the \( l \)th component of \( b_i \) affects only the \( l \)th component of \( \Gamma_i \). Noting that \( \Gamma_{i,l} \) is (weakly) increasing in \( b_{il} \), we have \( \Gamma_{i,l}(b_{il}|Z_{-i}) - \Gamma_{i,l}(b_{il} + \epsilon|Z_{-i}) \leq 0 \). Furthermore, if bidder \( i \)'s probability of winning object \( l \) strictly increases as the \( l \)'s component of the bid vector changes from \( b_{il} \) to \( b_{il} + \epsilon \), then

\[ v_{il} \leq -\frac{(P_i(b_i|Z_{-i}) - P_i(b_i + \epsilon|Z_{-i}))^T \kappa(Z_i, W_i)}{\Gamma_{i,l}(b_{il}|Z_{-i}) - \Gamma_{i,l}(b_{il} + \epsilon|Z_{-i})} + \]

\[ \frac{b_{il} \Gamma_i(b_{il}|Z_{-i}) - (b_{il} + \epsilon) \Gamma_i(b_{il} + \epsilon|Z_{-i})}{\Gamma_{i,l}(b_{il}|Z_{-i}) - \Gamma_{i,l}(b_{il} + \epsilon|Z_{-i})}. \] (38)

If \( \Gamma_{i,l}(b_{il}|Z_{-i}) - \Gamma_{i,l}(b_{il} + \epsilon|Z_{-i}) = 0 \), then \( P_i(b_i|Z_{-i}) - P_i(b_i + \epsilon|Z_{-i}| = 0 \) and we obtain the following inequality that clearly holds:

\[ 0 \geq -\epsilon \Gamma_i(b_{il}|Z_{-i}). \]

54
If there exists a known scalar $\bar{v} < \infty$ such that $V_{il} \leq \bar{v}$ with probability 1 for any $l$ (note that $\bar{v}$ could be strictly outside the support of $V_{il}$), then in this situation we can just bound $v_{il}$ from above by $\bar{v}$.

Analogously, taking $b = b_i - \epsilon e_l \in B_i$ for $\epsilon > 0$, we obtain from (37) that

$$v_{il}(\Gamma_{i,l}(b_{il}|Z_{-i}) - \Gamma_{i,l}(b_{il} - \epsilon|Z_{-i})) + (P_i(b_i|Z_{-i}) - P_i(b_i + \epsilon e_l|Z_{-i}))^T \kappa(Z_i, W_i) \geq b_{il}\Gamma_i(b_{il}|Z_{-i}) - (b_{il} - \epsilon)\Gamma_i(b_{il} - \epsilon|Z_{-i}).$$

If bidder $i$’s probability of winning object $l$ strictly increases as the $l$th component of the bid vector changes from $b_{il} - \epsilon$ to $b_{il}$, then

$$v_{il} \geq -\frac{(P_i(b_i|Z_{-i}) - P_i(b_i + \epsilon e_l|Z_{-i}))^T \kappa(Z_i, W_i)}{\Gamma_{i,l}(b_{il}|Z_{-i}) - \Gamma_{i,l}(b_{il} - \epsilon|Z_{-i})} + \frac{b_{il}\Gamma_i(b_{il}|Z_{-i}) - (b_{il} - \epsilon)\Gamma_i(b_{il} - \epsilon|Z_{-i})}{\Gamma_{i,l}(b_{il}|Z_{-i}) - \Gamma_{i,l}(b_{il} - \epsilon|Z_{-i})}. \quad (39)$$

If $\Gamma_{i,l}(b_{il}|Z_{-i}) - \Gamma_{i,l}(b_{il} - \epsilon|Z_{-i}) = 0$, then $P_i(b_i|Z_{-i}) - P_i(b_i - \epsilon e_l|Z_{-i}) = 0$ and we obtain the inequality $0 \geq -\epsilon\Gamma_i(b_{il}|Z_{-i})$, which implies that $\Gamma_i(b_{il}|Z_{-i}) = 0$. If there exists a known scalar $\bar{v} \geq 0$ such that $V_{il} \geq \bar{v}$ with probability 1 for any $l$ (note that $\bar{v}$ could be strictly outside the support of $V_{il}$), then in this situation we can just bound $v_{il}$ from below by $\bar{v}$.

Inequalities (38) and (39) will be the basis for our analysis. But before we proceed let us introduce some notations. Let $\Delta_{+}^{+}[f(u)]$ and $\Delta_{-}^{-}[f(u)]$ denote differences in the values of $f(\cdot)$ associated with adding $\epsilon$ and $-\epsilon$ to the $l$th component of $u$ respectively:

$$\Delta_{+}^{+}[f(u)] = f(u + \epsilon e_l) - f(u),$$

$$\Delta_{-}^{-}[f(u)] = f(u - \epsilon e_l) - f(u),$$

where $e_l$ denotes the $L$-dimensional $m$th unit vector.

For each $b_i \in B_i$, define $I_{\epsilon,l}^{+}(b_i|Z_{-i}), I_{\epsilon,l}^{-}(b_i|Z_{-i})$ as follows:

$$I_{\epsilon,l}^{+}(b_i|Z_{-i}) = \begin{cases} \bar{v} & \text{if } \Delta_{\epsilon,l}^{+}[\Gamma_i(b_i|Z_{-i})] = 0, \\ \Delta_{\epsilon,l}^{+}[b_i^T \Gamma_i(b_i|Z_{-i})] & \text{else} \end{cases}.$$  

$$I_{\epsilon,l}^{-}(b_i|Z_{-i}) = \begin{cases} \bar{v} & \text{if } \Delta_{\epsilon,l}^{-}[\Gamma_i(b_i|Z_{-i})] = 0, \\ \Delta_{\epsilon,l}^{-}[b_i^T \Gamma_i(b_i|Z_{-i})] & \text{else} \end{cases}.$$  

Also, for each $b_i \in B_i$, define the following $S_{\epsilon,l}^{-}(b_i|Z_{-i})$ and $S_{\epsilon,l}^{+}(b_i|Z_{-i})$:

$$S_{\epsilon,l}^{-}(b_i|Z_{-i}) = \begin{cases} 0 & \text{if } \Delta_{\epsilon,l}^{-}[\Gamma_i(b_i|Z_{-i})] = 0, \\ \Delta_{\epsilon,l}^{-}[P_i(b_i|Z_{-i})] & \text{else} \end{cases};$$  

$$S_{\epsilon,l}^{+}(b_i|Z_{-i}) = \begin{cases} 0 & \text{if } \Delta_{\epsilon,l}^{+}[\Gamma_i(b_i|Z_{-i})] = 0, \\ \Delta_{\epsilon,l}^{+}[P_i(b_i|Z_{-i})] & \text{else} \end{cases}.$$

For $K \in \mathcal{K}_i$, let $F_{il}^{-}(\cdot|K; Z_{-i})$ denote the c.d.f. of

$$\sup_{\epsilon > 0} \left(I_{\epsilon,l}^{-}(b_i|Z_{-i}) - S_{\epsilon,l}^{-}(b_i|Z_{-i})^T K\right),$$

55
and let \( \tilde{F}_{il}^+(\cdot|K;Z_{-i}) \) denote the c.d.f. of
\[
\inf_{\epsilon>0} \left( I_{\epsilon,l}^+(b_i|Z_{-i}) - S_{\epsilon,l}^+(b_i|Z_{-i})^T K \right)
\].

Then inequalities (38) and (39) imply that a superset of the identified set of \( \kappa(Z_i,W_i) \) for bidder \( i \) can be found as
\[
\bigcap_{m=1}^{L_i} \tilde{K}_{i,l}(Z_i, W_i),
\]
where \( \tilde{K}_{i,l}(Z_i, W_i) \) is defined as
\[
\tilde{K}_{i,l}(Z_i, W_i) = \{K \in \mathcal{K}_i \mid \tilde{F}_{il}^+(\cdot|K;Z_{-i}) \leq \tilde{F}_{il}^-(\cdot|K;Z'_{-i}) \quad \forall Z_{-i}, Z'_{-i} \in Z_{-i}|Z_i, W, X \}.
\]

Let us denote this superset as \( \mathcal{H}_{i,K}(Z_i, W_i) \).

Now we can construct supersets of the identified sets for the distributions of standalone valuations. As \( \mathcal{F}_c(\mathbb{R}^p) \) we denote the set of all continuous cumulative distribution functions on \( \mathbb{R}^p \).

A superset of the identified set for the c.d.f. of the standalone valuation \( V_{il} \) conditional on \( Z_i, X \) can be found as the set of univariate functions \( F_{il}(\cdot) \in \mathcal{F}_c(\mathbb{R}) \) such that for any \( \eta \in \mathbb{R} \),
\[
F_{il}(\eta) \in \bigcap_{W \in \mathcal{W}|Z_i,X} \bigcap_{\kappa_0 \in \mathcal{H}_{i,K}^{(1)}(Z_i,W_i)Z_{-i}} \bigcap_{Z'_{-i} \in Z_{-i}|Z_i,W,X} [\tilde{F}_{il}^+(\eta|\kappa_0;Z_{-i}), \tilde{F}_{il}^-(\eta|\kappa_0;Z'_{-i})].
\] (40)

Here we applied the exclusion restriction that the distribution of standalone valuations conditional on \( Z_i, W_i, X \) does not depend on \( W_i \). Let us denote this superset as \( \mathcal{H}_{i,K}^{(1)}(Z_i, X) \).

Our final step is to construct a superset \( \mathcal{H}_{i,F}^{(1)}(Z_i, X) \) for the identified set for the joint distribution of the vector of standalone valuations. \( \mathcal{H}_{i,F}^{(1)}(Z_i, X) \) can be found as the set of \( L_i \)-variate functions \( F_{il}(\cdot) \in \mathcal{F}_c(\mathbb{R}^{L_i}) \) such that each \( l \)th marginal distribution function generated by \( F_{il}(\cdot) \) belongs to \( \mathcal{H}_{i,F_l}^{(1)}(Z_i, X) \), \( l = 1, \ldots, L_i \). Moreover, for any \( \eta = (\eta_1, \ldots, \eta_{L_i}) \),
\[
F_{i}(\eta) \leq \min_{l=1,\ldots,L_i} \inf_{W \in \mathcal{W}|Z_i,X} \inf_{\kappa_0 \in \mathcal{H}_{i,F_l}^{(1)}(Z_i,W_i)} \inf_{Z_{-i} \in Z_{-i}|Z_i,W,X} \tilde{F}_{il}^+(\eta|\kappa_0;Z_{-i}),
\] (41)
\[
F_{i}(\eta) \geq \max \left\{ \sum_{m=1}^{L_i} \sup_{W \in \mathcal{W}|Z_i,X} \sup_{\kappa_0 \in \mathcal{H}_{i,F_l}^{(1)}(Z_i,W_i)} \sup_{Z_{-i} \in Z_{-i}|Z_i,W,X} \tilde{F}_{il}^-(\eta|\kappa_0;Z_{-i}) - L_i + 1, 0 \right\},
\] (42)

where we employed the well known result on sharp Fréchet-Hoeffding bounds for joint distributions.

We next provide an expectations version of the partial identification argument. Even though the supersets in the expectations approach will be larger than those discussed previously, computationally they are easier to obtain. Before describing these supersets, let us define \( L_i \times 1 \) vectors.
\( \bar{\Upsilon}_\epsilon^{-}(Z_{-i}), \bar{\Upsilon}_\epsilon^{+}(Z_{-i}) \) and \( L_i \times 2^{L_i} \) matrices \( \bar{\Psi}_\epsilon^{-}(Z_{-i}), \bar{\Psi}_\epsilon^{+}(Z_{-i}) \) as follows:
\[
\bar{\Upsilon}_\epsilon^{-}(Z_{-i}) \equiv \left[ E[I_i^{-}(B_i|Z_{-i})|Z_{-i}] \right]_{l=1}^{L_i} \\
\bar{\Upsilon}_\epsilon^{+}(Z_{-i}) \equiv \left[ E[I_i^{+}(B_i|Z_{-i})|Z_{-i}] \right]_{l=1}^{L_i} \\
\bar{\Psi}_\epsilon^{-}(Z_{-i}) \equiv \left[ E[S_i^{-}(B_i|Z_{-i})|Z_{-i}] \right]_{l=1}^{L_i} \\
\bar{\Psi}_\epsilon^{+}(Z_{-i}) \equiv \left[ E[S_i^{+}(B_i|Z_{-i})|Z_{-i}] \right]_{l=1}^{L_i}.
\]

Then, applying the expectation over the distribution of bids conditional on \( Z_i, W, X \) to inequalities (38) and (39) and pooling restrictions across \( Z_{-i}, Z'_{-i} \) and \( l = 1, \ldots, L_i \), we establish that a superset of the identified set for \( \kappa(Z_i, W_i) \) can be found in the following way:
\[
\mathcal{H}^{(2)}_{i, \kappa}(Z_i, W_i) = \bigcap_{\epsilon > 0} \bar{K}_\epsilon^{(Z_i, W_i)},
\]
where \( \bar{K}_\epsilon^{(Z_i, W_i)} \) is defined as
\[
\bar{K}_\epsilon^{(Z_i, W_i)} \equiv \left\{ K \in \mathcal{K}_i \mid \left( \bar{\Upsilon}_\epsilon^{-}(Z_{-i}) - \bar{\Upsilon}_\epsilon^{+}(Z'_{-i}) \right) - \left( \bar{\Psi}_\epsilon^{-}(Z_{-i}) - \bar{\Psi}_\epsilon^{+}(Z'_{-i}) \right) \leq 0 \text{ for all } Z_{-i}, Z'_{-i} \in Z_{-i}|Z_i, W, X \right\}.
\]

Notice two features of \( \mathcal{H}^{(2)}_{i, \kappa}(Z_i, W_i) \). First, it can be represented as the intersection of a set of half-spaces in \( \mathcal{K}_i \), where half-spaces are bounded by hyperplanes involving slope vectors \( (\bar{\Upsilon}_\epsilon^{-}(Z_{-i}) - \bar{\Psi}_\epsilon^{+}(Z'_{-i})) \) and intercepts \( (\bar{\Upsilon}_\epsilon^{-}(Z_{-i}) - \bar{\Psi}_\epsilon^{-}(Z'_i)) \), and the intersection is taken over collections of \( (Z_{-i}, Z'_{-i}, \epsilon, m) \).

Second, if \( G_i \) is absolutely continuous, then \( \mathcal{H}^{(2)}_{i, \kappa}(Z_i, W_i) \) is a singleton, and as we show below, the analysis of \( \mathcal{H}^{(2)}_{i, \kappa}(Z_i, W_i) \) essentially becomes our identification strategy in the case of point identification. Indeed, bidder \( i \)'s objective function is differentiable at almost every observed \( b_i \). Hence as \( \epsilon \to 0 \) we have for all \( l \)
\[
\lim_{\epsilon \to 0} \frac{\Delta^\epsilon_{-}\Gamma_{i}(b_i|Z_{-i})}{\Delta^\epsilon_{-}\Gamma_{i}(b_i|Z_{-i})} = \lim_{\epsilon \to 0} \frac{\Delta^\epsilon_{-}\Gamma_{i}(b_i|Z_{-i})}{\Delta^\epsilon_{-}\Gamma_{i}(b_i|Z_{-i})} = \frac{\partial(b_i\Gamma_{i}(b_i|Z_{-i}))}{\partial b_{il}},
\]
and therefore \( \bar{\Upsilon}_\epsilon^{-}() \to \bar{\Upsilon}(\cdot) \). Analogously, it is straightforward to show that \( \bar{\Upsilon}_\epsilon^{+}() \to \bar{\Psi}(\cdot) \), and \( \bar{\Psi}_\epsilon^{-} \to \bar{\Psi}(\cdot) \). Hence after applying the expectations operator, inequalities (38) and (39) imply that
\[
\bar{\Upsilon}(Z_{-i}) - \bar{\Psi}(Z_{-i}) \kappa_0 \leq \bar{\Upsilon}(Z'_{-i}) - \bar{\Psi}(Z'_{-i}) \kappa_0 \quad \forall \ Z_{-i}, Z'_{-i} \in Z_{-i}|Z_i, W, X.
\]
Noting that \( Z_{-i}, Z'_{-i} \) are interchangeable, we thus have for any \( Z_{-i}, Z'_{-i} \in Z_{-i}|Z_i, W, X \):
\[
\bar{\Upsilon}(Z_{-i}) - \bar{\Psi}(Z_{-i}) \kappa_0 \leq \bar{\Upsilon}(Z'_{-i}) - \bar{\Psi}(Z'_{-i}) \kappa_0 \\
\bar{\Upsilon}(Z'_{-i}) - \bar{\Psi}(Z'_{-i}) \kappa_0 \leq \bar{\Upsilon}(Z_{-i}) - \bar{\Psi}(Z_{-i}) \kappa_0,
\]
or equivalently
\[
\bar{\Upsilon}(Z_{-i}) - \bar{\Psi}(Z_{-i}) \kappa_0 = \bar{\Upsilon}(Z'_{-i}) - \bar{\Psi}(Z'_{-i}) \kappa_0 \quad \forall \ Z_{-i}, Z'_{-i} \in Z_{-i}|Z_i, W, X.
\]

57
But this is exactly the identification restriction invoked in Proposition 3 in the main text. Thus we can strictly generalize our existing identification results (which depend on differentiability a.e.) to partial identification for arbitrary $G_i$.

A superset for the identification set of the c.d.f. of $V_{it}$ can be found as in (40) by replacing $\mathcal{H}_{i,\kappa}^{(1)}(Z_i, W, X)$ with $\mathcal{H}_{i,\kappa}^{(2)}(Z_i, W, X)$. Similarly, a superset for the identification set of the c.d.f. of vector $V_i$ can be found as in (41) and (42) by replacing $\mathcal{H}_{i,\kappa}^{(1)}(Z_i, W, X)$ with $\mathcal{H}_{i,\kappa}^{(2)}(Z_i, W, X)$.

Appendix F: Nonparametric tests for complementarities

Finally, we briefly discuss approaches to testing for the presence of complementarities based on the bid data together with bidder-specific and individual-specific characteristics. Taking the complementarities function as deterministic, the condition of the absence of complementarities can be written as

$$\kappa_i(Z, W_i) = 0$$

for each $i$.

We can rely on testing the following testable implications on the distribution of bids:

$$H_0 : E[B_i | X, Z, W] = E[B_i | X, Z] \quad a.e.,$$

which can be equivalently written as

$$E[B_i - E[B_i | X, Z] | X, Z, W] = 0 \quad a.e.$$  

The alternative hypothesis $H_1$ is the negation of the null.

There are a variety of tests in econometrics and statistics for testing the null hypothesis above. We could, e.g., use the test of Delgado and Manteiga (2003), which requires residuals obtained from the nonparametric estimation under the null. We could also employ tests in Lavergne (2001) or Neumeyer and Dette (2003) that develop general tests for the equality of two nonparametric regression curves (the former work uses smoothing techniques whereas the latter is non-smoothing test). A test proposed in Racine (1997) looks as whether the partial derivatives of the regression function with respect to the variables being tested are zero.

Many of these tests require the assumption of i.i.d. observations. In light of this, we can focus only on the bidders that participate in two (or some other fixed number of auctions) auctions and construct test statistics by picking one such bidder per letting date.

References


