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by

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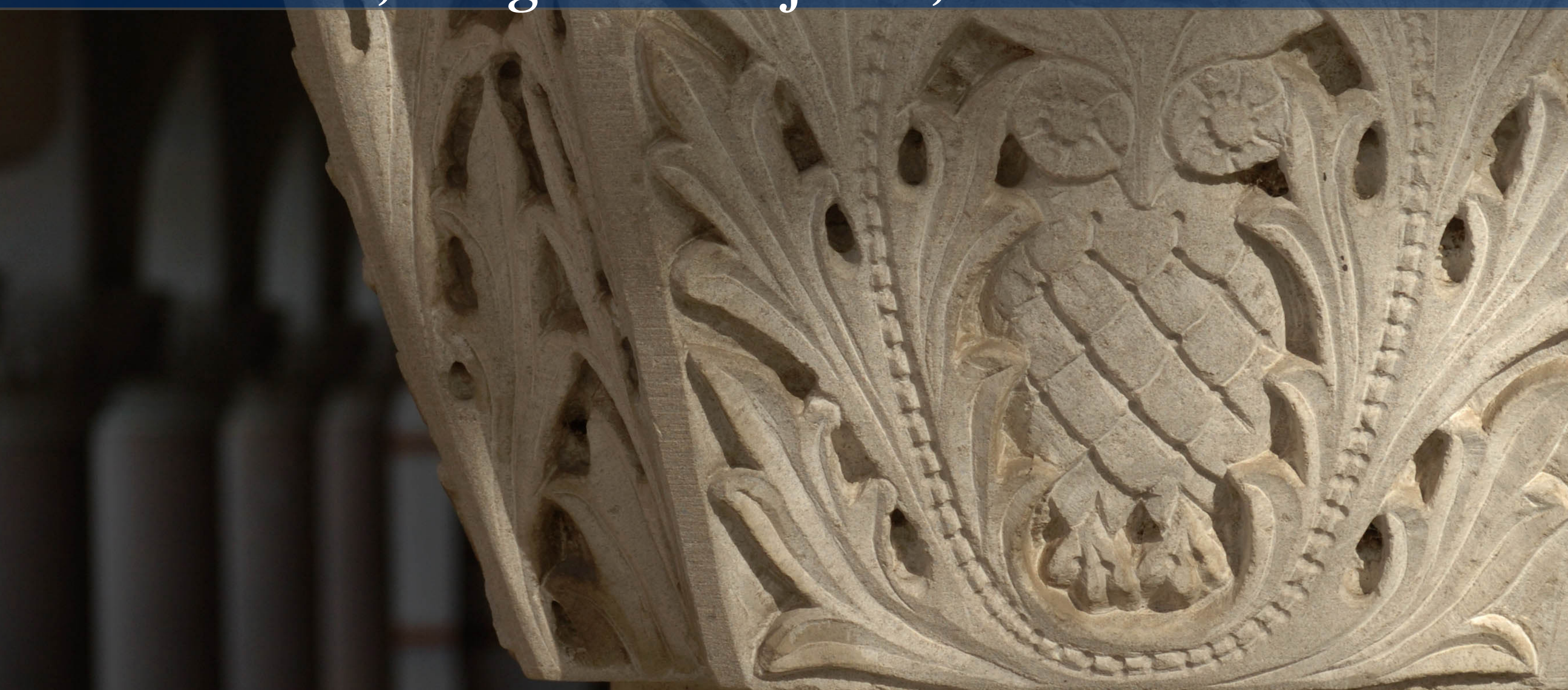

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Robust Optimal Taxation and Environmental Externalities*

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Abstract

We study optimal taxation in a dynamic stochastic general equilibrium model where agents are concerned about model uncertainty regarding climate change. An externality from greenhouse gas emissions adversely affects the economy's capital stock. We assume that the mapping from climate change to damages is subject to *uncertainty*, and we adapt and use techniques from robust control theory in order to study efficiency and optimal policy. We obtain a sharp analytical solution for the implied environmental externality and we characterize dynamic optimal taxation. A small increase in the concern about model uncertainty can cause a significant drop in optimal energy extraction. The optimal tax which restores the social optimal allocation is Pigouvian. Under more general assumptions, we develop a recursive method and solve the model computationally. We find that the introduction of uncertainty matters qualitatively and quantitatively. We study optimal GDP growth in the presence and in the absence of concerns about uncertainty and find that these can lead to different conclusions.

*Comments welcome.

1 Introduction

We study optimal taxation in a dynamic stochastic general equilibrium model where agents are concerned about model uncertainty.[1] We assume that an externality through global temperature-changes from Green House Gas emissions *(GHG)* adversely affects the economy's capital stock and, thus, output. Its precise effects, however, are subject to uncertainty. In order to model the effect of the emissions created by economic activity on the environment, we use the framework in Golosov, Hassler, Krusell, and Tsyvinski (GHKT, 2012).[2] While they assume that the mapping from climate change to damages is subject to *risk*, in our model this mapping is subject to *Knightian uncertainty.* We study the implications of this assumption using a robust control approach. We believe that this is an appropriate application of uncertainty in economic modeling. After all, man-made climate change is unprecedented, and there is an ongoing heated debate regarding its potential effects. More specifically, concerned about model uncertainty, a social planner in our model maximizes social welfare under a "worst-case scenario."

Unlike GHKT (2012), we assume that the environmental externality affects output indirectly, through the capital stock. As a result, the theoretical analysis in our model brings different results, although the two models are identical in that respect if we assume 100% capital depreciation (as we do in the computational part). Another important difference, in addition to taking into consideration model-uncertainty, is that we use estimates about total oil and gas supplies that are larger than theirs. This is partly due to adding the supply of unconventional oil and gas, but mainly due to considering methane hydrates.[3]

Under plausible assumptions, we obtain a sharp analytical solution for the implied pollution externality and we characterize dynamic optimal taxation. A small increase in the concern about model uncertainty can cause a significant drop in optimal energy extraction. The optimal tax, which restores the social optimal allocation, is Pigouvian. Under more general assumptions, we develop a simple recursive method that allows us to solve the model computationally. We find that the introduction of uncertainty matters, in the sense that our model produces results that are qualitative different, for example, in terms of oil consumption, from GHKT (2012). At the same time, concerns about uncertainty do not affect renewable energy adoption. The reason is that the margin that determines short-term decisions regarding energy sources is driven by two factors: the trade-off between higher versus lower total energy consumption, and the choice of coal versus gas/oil, rather than by renewable energy use. We find that oil-use in our model can be flat for some parametrizations. We

[1] Hansen and Sargent (2008) provide several examples of applications of these methods in economics. In a recent paper, Bidder and Smith (2012) use robust control theory to study the implications of model uncertainty for business cycles generated through "animal spirits." Our work involves analyzing a class of robust control problems outside the standard Linear-Quadratic specification.

[2] Acemoglu, Aghion, Bursztyn, and Hemous (2012) study related issues. See Nordhaus and Boyer (2000) and Stern (2007) for earlier work that also points to the importance of uncertainty.

[3] This allows for robustness considerations to have bite in our model. See Hartley, Medlock, Temzelides, and Zhang (2012) and Adao, Narajabad, and Temzelides (2012) and references therein for a more detailed discussion on total estimated fossil fuel resources.

study optimal GDP growth in the presence and in the absence of concerns about uncertainty, and find that the results can be very different. In the worst case scenario, optimality implies that a small sacrifice in yearly GDP can avoid a large future welfare loss.

Since the green energy sector does not create emissions, we find that the optimal path for the use of green energy does not directly depend on the level of concern about model uncertainty. However, since green energy, coal, and oil are substitutes, model uncertainty does affect the use of green energy indirectly, through its impact on coal and oil. We also find that an increase in the concern about model uncertainty causes a significant decline in the use of coal, while the use of oil is delayed, but only slightly. Holding other parameters fixed, the optimal path of oil consumption is determined jointly by the resource scarcity effect and by the model uncertainty effect. Naturally, we do not find a significant difference between the robust and non-robust optimal path of oil consumption when the scarcity effect dominates. However, when we consider a higher initial resources of fossil fuel, the concern about model uncertainty substantively discourages the use of oil.

The paper proceeds as follows. Section 2 presents the basic model. Section 3 studies the model analytically, while Section 4 presents our numerical and quantitative findings. A brief conclusion follows. The Appendix contains some technical material.

2 The Model

In order to characterize the optimal policy for dealing with climate change in the case where there is a concern about model uncertainty, we will build on a version of GHKT (2012). In this section, we formulate a general framework for the "robust planner's problem," a benchmark that we will subsequently compare to various decentralized market solutions.

Time, t, is discrete and the horizon is infinite. The world economy is populated by a $[0,1]$-continuum of infinite-lived representative agents with utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t) \tag{1}$$

The function u is a standard concave period utility function, C_t represents final-good consumption in period t, and $\beta \in (0,1)$ is the discount factor. The final goods sector uses energy, E, capital, K, and labor, N, to produce output. Labor supply is inelastic. The economy's capital stock depreciates at rate $\delta \in (0,1)$. Henceforth, $\tilde{K}$ represents the end-of-period capital (before interacting with the climate factor through a process described below). The feasibility constraint in the final goods sector is given by

$$C_t + \tilde{K}_{t+1} = Y_t + (1-\delta)K_t \tag{2}$$

There are four production sectors. The final-goods sector, indexed by $i = 0$, produces the consumption good using production function $Y = F(K, N_0, E)$. Thus, in addition to capital and labor, production of the final good requires the use of energy, E. The three energy-producing sectors (oil, coal, and green energy; labelled by $i = 1, 2, 3$, respectively) produce energy amounts E_1, E_2 and E_3 (measured in carbon equivalents). The oil sector is assumed to produce oil at zero cost. We denote by R the total oil energy stock and we impose the resource constraint, $R_t \geq 0$, all t. Both the coal and the green energy sectors use linear technologies

$$E_i = A_i N_i, \ i = 2, 3 \tag{3}$$

The variable S (measured in units of carbon content) represents the GHG concentration in the atmosphere in excess of the pre-industrial level. We denote by P and T the permanent and temporary components of S, respectively. These evolve according to the following.

$$P' = P + \phi_L(E_1 + E_2) \tag{4}$$

$$T' = (1-\phi)T + (1-\phi_L)\phi_0(E_1 + E_2) \tag{5}$$

$$S' = P' + T' \tag{6}$$

We introduce model uncertainty regarding climate change by introducing a stochastic variable, γ, which reduces the end-of-period capital stock $\tilde{K}'$ by a factor of $h(S', \gamma)$ to K'. That is, $K' = h(S', \gamma)\tilde{K}'$.[4] We use $\pi(\gamma)$ to denote the approximating distribution of γ, while $\hat{\pi}(\gamma)$ denotes the welfare-minimizing distribution, and $m(\gamma) = \frac{\hat{\pi}(\gamma)}{\pi(\gamma)}$ is the likelihood ratio. Moreover, the distance, ρ, between $\hat{\pi}(\gamma)$ and $\pi(\gamma)$ is measured by relative entropy.

$$\rho(\hat{\pi}(\gamma), \pi(\gamma)) \equiv E[m(\gamma)\log m(\gamma)] \equiv \hat{E}[\log m(\gamma)] \equiv \int [m(\gamma)\log m(\gamma)]\pi(\gamma)\mathrm{d}\gamma \tag{7}$$

As is standard in robust control theory, the concern about model uncertainty is represented by a two-person zero-sum dynamic game in which, after observing the choice of a social planner, a malevolent player chooses the worst specification of the model in each period. This game proceeds as follows.[5] At the beginning of a period, the state; i.e., the value of (K, N, P, T, R) is revealed. Then, the planner moves, choosing $(C, E_i, N_i, \tilde{K}', P', T', S', R')$ in order to maximize social welfare. After observing the planner's choice, nature (or the "malevolent player") chooses an alternative distribution $\hat{\pi}(\gamma)$ or, equivalently, $m(\gamma)$, to minimize welfare. Note that any deviation from the approximating distribution will be penalized by adding $\alpha\rho(\hat{\pi}(\gamma), \pi(\gamma))$ to the objective function. Here, α represents the magnitude of the "punishment." A greater α means a greater penalty associated with the deviation of γ from its approximating distribution, thus, a lower concern about robustness.

[4] In GHKT (2012), γ directly affects output. We find it convenient to assume that γ adversely affects the economy's capital stock. The two assumptions lead to identical results when there is 100% capital depreciation (which we assume for our numerical results).

[5] Our attention will be restricted to a particular type of equilibrium, the so-called Markov perfect (or *feedback*) equilibrium. This equilibrium is *strongly time-consistent*.

This results in the following social planner's problem:

$$V(K,N,P,T,R) = \max_{\{C,E_i,N_i,\tilde{K}',P',T',S',R'\}} \min_{m(\gamma)}$$
$$\{u(C) + \beta \int [m(\gamma)V(K',N',P',T',R') + \alpha m(\gamma) \log m(\gamma)] \pi(\gamma) \mathrm{d}\gamma\} \tag{8}$$

$s.t.$

$$E_i = A_i N_i;\ i = 2,3 \tag{9}$$
$$E = (\kappa_1 E_1^\rho + \kappa_2 E_2^\rho + \kappa_3 E_3^\rho)^{1/\rho} \tag{10}$$
$$N = N_0 + N_2 + N_3 \tag{11}$$
$$\tilde{K}' = F(K, N_0, E) + (1-\delta)K - C \tag{12}$$
$$K' = h(S',\gamma)\tilde{K}' \tag{13}$$
$$R' = R - E_1 \geq 0 \tag{14}$$
$$N' = A_N N \tag{15}$$
$$P' = P + \phi_L(E_1 + E_2) \tag{16}$$
$$T' = (1-\phi)T + (1-\phi_L)\phi_0(E_1+E_2) \tag{17}$$
$$S' = P' + T' \tag{18}$$
$$1 = \int m(\gamma)\pi(\gamma)\mathrm{d}\gamma \tag{19}$$

The social planner's problem can be solved analytically under a set of additional assumptions. We will first focus on the analytical solution. We will discuss the decentralized problem and show that the socially optimal allocation can be restored by implementing fossil fuel taxes on the energy-producing sector.

3 The Analytical Solution

For the remainder of this section, we will make the following assumptions. While these assumptions are admittedly strong, they allow us to fully solve the model analytically. As we shall see, certain aspects of the analytical solution will remain instructive in the next section, when these assumptions are dropped and the model is solved numerically.

(A1) The utility function is given by $u(C) = \log(C)$.

(A2) Capital depreciates fully; i.e., $\delta = 1$.

(A3) The production function is given by $F(K, N_0, E) = A_0 K^\theta N_0^{1-\theta-\nu} E^\nu$.

(A4) The damage function is given by $h(S',\gamma) = e^{-S'\gamma}$.[6]

(A5) The approximating distribution for γ is exponential with mean λ^{-1} and variance λ^{-2}; i.e., $\pi(\gamma) = \lambda e^{-\lambda\gamma}$.[7]

[6] Note that $\frac{1}{\Delta}$ is a threshold for S. If $S \geq \frac{1}{\Delta}$, the system cannot be "robustified," in the sense that the value of the game goes to negative infinity. However, if the economy starts with an initial $S_0 < \frac{1}{\Delta}$, then S_t will converge to $\frac{1}{\Delta}$ as $t \to +\infty$.

[7] The exponential distribution with mean λ^{-1} is the maximum entropy distribution among all continuous

(A6.1) $\phi_L = 0$.[8]

(A6.2) $\phi = 0$.

(A7) There is a single fossil energy sector producing oil at zero cost. Production is subject to a resource feasibility constraint: $R' \geq 0$. As a result, $N_1 = 0$ and $N_0 = N$.

(A8) There is no population growth and the aggregate labor supply is normalized to be 1. That is, $A_N = 1$ and $N = 1$ in all periods.

(A9) There is no technology improvement. That is, A_0 is constant over time. We normalize $A_0 = 1$.

(A10) The resource feasibility constraint is not binding.[9]

We will first solve the social planner's problem. We will then discuss the decentralized problem and show that the socially optimal allocation can be restored by implementing fossil fuel taxes on the energy-producing sector.

Under **A1-A10**, the social planner's problem can be rewritten as:

$$\begin{aligned}
& V(K,S) \\
= \; & \max_{\{C,E,\tilde{K}',S'\}} \min_{m(\gamma)} \{u(C) + \beta \int [m(\gamma)V(K',S') + \alpha m(\gamma) \log m(\gamma)] \, \pi(\gamma) \mathrm{d}\gamma\} \quad (20) \\
& s.t. \\
\tilde{K}' = \; & F(K,E) - C \quad (21) \\
K' = \; & h(S',\gamma)\tilde{K}' \quad (22) \\
S' = \; & S + \phi_0 E \quad (23) \\
1 = \; & \int m(\gamma)\pi(\gamma)\mathrm{d}\gamma \quad (24)
\end{aligned}$$

where $h(S',\gamma) = e^{-S'\gamma}$ and $F(K,E) = K^{\theta}E^{\nu}$. To solve this problem, we first guess that $V(\cdot)$ takes the form

$$V(K',S') = f(S') + \bar{A}\log(K') + \bar{D} = f(S') + \bar{A}\log(h(S',\gamma)\tilde{K}') + \bar{D} \quad (25)$$

where $\bar{A}$ and $\bar{D}$ are undetermined coefficients. The functional form for $f(\cdot)$ will be derived when we solve the minimizing player's problem.

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distributions supported in $[0,\infty]$ that have mean λ^{-1}. The worst case distribution for γ is also exponential with mean $(\lambda^*)^{-1}$ and variance $(\lambda^*)^{-2}$, where $\lambda^* = \lambda(1 - \Delta S'^*) = \lambda(1 - \Delta\phi_0 c_E)(1 - \Delta S)$. That is, $\pi^*(\gamma) = \lambda^* e^{-\lambda^*\gamma}$. Since $\lambda^* = \lambda(1 - \Delta S'^*) < \lambda$, the worst case mean of γ, $(\lambda^*)^{-1}$, is strictly greater than the approximating mean, λ^{-1}.

[8] If $\phi_L > 0$, we need to depict the dynamics of P and T separately before we add them up in order to obtain the dynamics of S. Assuming that $\phi_L = 0$, allows us to express the dynamics of S without the need to consider P and T separately. That is, $S' = (1-\phi)S + \phi_0 E$. Moreover, under **(A6.1)** and **(A6.2)**, we have $S' = S + \phi_0 E$, which is necessary for an analytical solution.

[9] Later we provide a sufficient condition for **(A10)**.

First, we define the robustness problem (the *inner minimization problem*) by

$$\begin{aligned}
\mathcal{R}(V)(\tilde{K}', S') &= \min_{m(\gamma)} \int [m(\gamma)V(K', S') + \alpha m(\gamma)\log m(\gamma)]\,\pi(\gamma)\mathrm{d}\gamma && (26)\\
s.t. & \\
& K' = e^{-S'\gamma}\tilde{K}' && (27)\\
1 &= \int m(\gamma)\pi(\gamma)\mathrm{d}\gamma && (28)
\end{aligned}$$

The F.O.N.C. for $m(\gamma)$ implies that

$$m^*(\gamma) = \frac{\exp(-\frac{V(K',S')}{\alpha})}{\int \exp(-\frac{V(K',S')}{\alpha})\pi(\gamma)\mathrm{d}\gamma} = (1-\Delta S')e^{\Delta S'\lambda\gamma} \tag{29}$$

or equivalently,

$$\hat{\pi}^*(\gamma) = m^*(\gamma)\pi(\gamma) = \lambda^* e^{-\lambda^*\gamma} \tag{30}$$

where we define $\Delta = \frac{\bar{A}}{\alpha\lambda}$and $\lambda^* = \lambda(1-\Delta S')$.[10] Thereby,

$$\begin{aligned}
\mathcal{R}(V)(\tilde{K}', S') &= \int [m^*(\gamma)V(K', S') + \alpha m^*(\gamma)\log m^*(\gamma)]\,\pi(\gamma)\mathrm{d}\gamma \\
&= -\alpha\log[\int \exp(-\frac{V(K',S')}{\alpha})\pi(\gamma)\mathrm{d}\gamma] && (31)
\end{aligned}$$

Substituting equation(**??**) into equation(31), we obtain

$$\mathcal{R}(V)(\tilde{K}', S') = f(S') + \bar{A}\log(\tilde{K}') + \bar{D} + H(S';\alpha,\bar{A}) \tag{32}$$

where $H(S';\alpha,\bar{A})$, the robust version of the externality from carbon emissions, is given by

$$H(S';\alpha,\bar{A}) = -\alpha\log[\int h^{-\frac{\bar{A}}{\alpha}}(S',\gamma)\pi(\gamma)\mathrm{d}\gamma] \tag{33}$$

It follows from **(A4)**-**(A5)** that

$$H(S';\alpha,\bar{A}) = \alpha\log(1-\Delta S') \tag{34}$$

Next, we define the optimal choice problem (the *outer maximization problem*). Using the above result, this problem can be written as

$$V(K,S) = \max_{\{C,E,\tilde{K}',S'\}} \{\log(C) + \beta\mathcal{R}(V)(\tilde{K}', S')\} \tag{35}$$

[10] The worst case distribution of γ remains exponential with a distorted mean $(\lambda^*)^{-1}$ and variance $(\lambda^*)^{-2}$.

or equivalently,

$$\begin{aligned} & f(S) + \bar{A}\log(K) + \bar{D} \\ = & \max_{C,E}\{\log(C) + \beta[f(S') + \bar{A}\log(\tilde{K}') + \bar{D} + H(S';\alpha,\bar{A})]\} \end{aligned} \tag{36}$$

$$s.t.$$

$$\tilde{K}' = F(K,E) - C \tag{37}$$

$$S' = S + \phi_0 E \tag{38}$$

$$H(S';\alpha,\bar{A}) = \alpha\log(1-\Delta S') \tag{39}$$

The F.O.N.C. imply

$$C = \frac{F(K,E)}{1+\beta\bar{A}} \tag{40}$$

$$-\phi_0\left[\frac{\partial f(S')}{\partial S'} + \frac{\partial H(S';\alpha,\bar{A})}{\partial S'}\right] = \frac{\beta\bar{A}+1}{\beta}\frac{\frac{\partial F(K,E)}{\partial E}}{F(K,E)} \tag{41}$$

Noting that $H(S;\alpha,\bar{A})$ is a logarithmic function of S, we guess that $f(S) = \bar{B}\log(1-\Delta S)$, where $\bar{B}$ is an undetermined coefficient. As a result, the above F.O.N.C. can be simplified to

$$C = \frac{K^\theta E^\nu}{1+\beta\bar{A}} \tag{42}$$

$$\frac{\beta\phi_0\Delta(\alpha+\bar{B})}{1-\Delta S'} = \frac{\nu(\beta\bar{A}+1)}{E} \tag{43}$$

After some tedious derivations, we obtain

$$\bar{A} = \frac{\theta}{1-\beta\theta} \tag{44}$$

$$\bar{B} = \frac{1}{1-\beta}[\alpha\beta + \frac{\nu}{1-\beta\theta}] \tag{45}$$

The expression for $\bar{D}$ is more complicated and less intuitive. Substituting $\bar{A} = \frac{\theta}{1-\beta\theta}$ into the F.O.N.C., we obtain the optimal allocation. Thus, we have the following.

Proposition 1. *Suppose assumptions (A1)-(A10) hold. Then the two-person zero-sum dynamic game described by eq(20)-eq(24) admits a feedback (Markov perfect) equilibrium. The equilibrium strategies are given by:*

$$C^* = (1-\beta\theta)K^\theta E^{*\nu} = (1-\beta\theta)K^\theta[c_E(1-\Delta S)]^\nu \tag{46}$$

$$E^* = c_E(1-\Delta S) \tag{47}$$

$$S'^* = S + \phi_0 c_E(1-\Delta S) \tag{48}$$

$$\hat{\pi}^*(\gamma) = \lambda^* e^{-\lambda^*\gamma} \tag{49}$$

where $c_E = \frac{\nu(1-\beta)}{[\beta\alpha(1-\beta\theta)+\nu]\phi_0\Delta}$.

A few technical remarks are in order. First, the function $V(K,S)$ is increasing in K, decreasing in S, and jointly concave in K and S. The value of $\bar{A}$ is the same as in the model without concern about model uncertainty. Both E^* and S'^* are affine functions of S. In addition, we can show that, given S, both of E^* and S'^* are increasing functions of α. This is intuitive since a greater α implies a larger penalty from a deviation of γ from its approximating distribution, thus, a lower concern about model-uncertainty. Note that C^* is affected by S only through E^*. This is due to logarithmic utility. As a result, a greater concern about model-uncertainty will lower both E^* and C^*. The value of the externality from one unit of emissions evaluated at E^* is given by

$$\lambda^s = -\beta \frac{\partial V(K', S')}{\partial E}|_{K'^*, S'^*} = \frac{\beta \phi_0 \Delta(\bar{B} + \alpha)}{1 - \Delta S'^*} = \frac{\nu}{c_E(1-\beta\theta)(1-\Delta S)} = \frac{\nu}{(1-\beta\theta)E^*} \tag{50}$$

Our model so far is similar to the oil regime in GHKT (2012), except that we assume that the resource constraint is not binding. Since $S_{t+1} = S_t + \phi_0 E_t$, we arrive at the following expression for the aggregate oil extraction

$$\sum_{t=0}^{+\infty} E_t = \lim_{t \to +\infty} \phi_0^{-1}(S_t - S_0) = \phi_0^{-1}(\frac{1}{\Delta} - S_0) \tag{51}$$

Thus, the resource constraint is not binding if and only if the aggregate oil reserves are greater than $\phi_0^{-1}(\frac{1}{\Delta} - S_0)$. Figures 1, 2, and 3 below illustrate how E^* responds to a concern about model-uncertainty. Figures 1 and 2 show how E^* reacts to a change in the penalty parameter, α, in the multiplier version of the game. Figure 3 refers to the equivalent constraint game, in which $\hat{\pi}(\gamma)$ is constrained in a closed ball of radius δ centered at $\pi(\gamma)$, denoted by $B_\delta(\pi(\gamma))$. Direct calculation shows that the distance between $\hat{\pi}^*(\gamma)$ and $\pi(\gamma)$, as measured by entropy is given by

$$\rho(\hat{\pi}^*(\gamma), \pi(\gamma)) = \log(1 - \Delta S'^*) + \frac{\Delta S'^*}{1 - \Delta S'^*} \tag{52}$$

Since $\hat{\pi}^*(\gamma)$, which is chosen by the minimizing player, must be on the boundary of $B_\delta(\pi(\gamma))$, we have that $\rho(\hat{\pi}^*(\gamma), \pi(\gamma)) = \delta$. Figure 3 shows how E^* changes as we relax δ, allowing for more uncertainty about the approximating model. In the Appendix we show that $\frac{\partial E^*}{\partial \delta}|_{\delta=0} = -\infty$. That is, even an infinitesimal concern about model uncertainty can cause a significant drop in the optimal energy extraction.

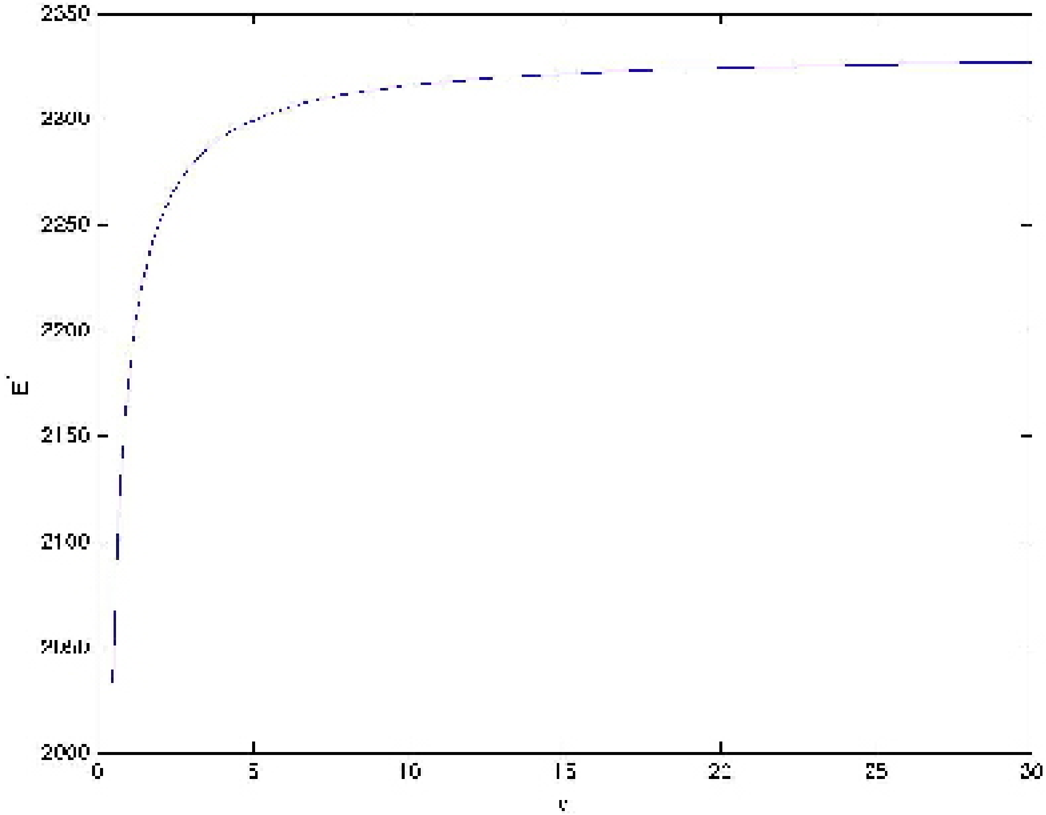

Figure 1: The Effect of Penalty Parameter α on Optimal Carbon Emissions, E

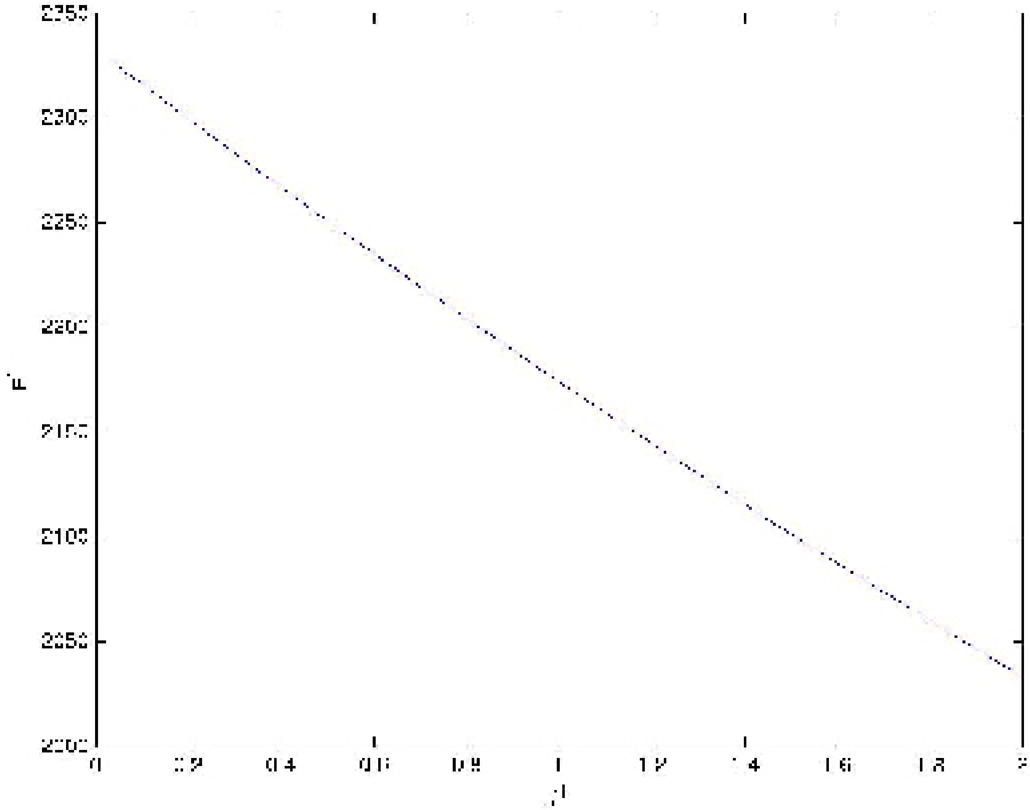

Figure 2: The Effect of α^{-1} on E

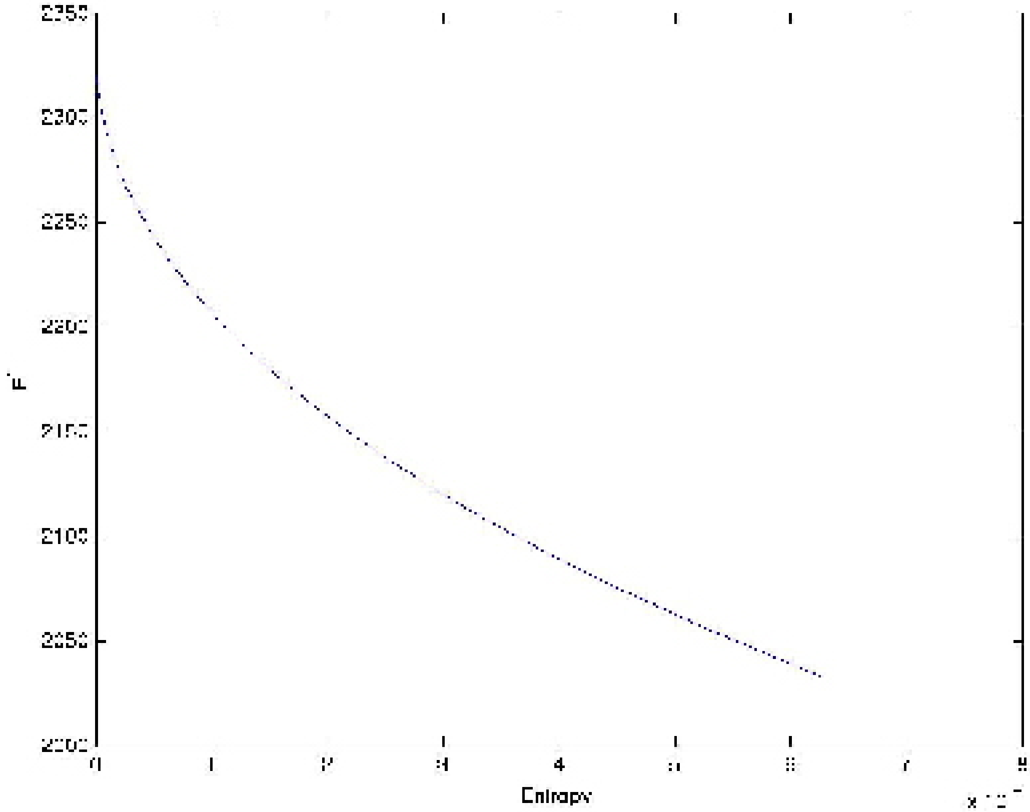

Figure 3: The Effect of Model Deviation as Measured by Entropy, δ, on E

Robust control modeling can be introduced in different ways. So far we used a closed-loop zero-sum dynamic game in which the social planner moves first in each period. Alternatively, we can construct a game with the same information structure by interchanging the order of max and min in eq(20). The two games differ only in terms of the timing protocol. However, both lead to the same unique feedback saddle-point equilibrium if certain conditions are satisfied. More precisely, if (A1)-(A10) hold, then the objective in (20) is strictly concave in C and E, and strictly convex in $m(\gamma)$. Consequently, the two closed-loop zero-sum dynamic games admit the same unique pure strategy saddle-point Nash equilibrium, which is the one given in Proposition 1.

3.1 Decentralization

Let us now turn to the decentralized problem. Suppose a percentage tax, τ_t, is imposed on emissions, E_t. Since the extraction cost of energy (thus, the cost of creating emissions) is zero, it must be true that

$$\tau_t = p_t = \frac{\partial F(K_t, E_t)}{\partial E_t} = \nu K_t^{\theta} E_t^{\nu-1} \tag{53}$$

The above equation captures the one-to-one relationship between E_t and τ_t. Therefore, to achieve the optimal emissions level, $E_t = c_E(1-\Delta S)$ in eq(47), we must impose $\tau_t = \nu c_E^{\nu-1}(1-\Delta S_t)^{\nu-1}K_t^\theta$. It is straightforward to show that $\tau_t = \frac{\lambda^s}{u'(C_t^*)}$, where C_t^* is the optimal consumption, given by eq(46). That is, the optimal tax on emissions is equal to the corresponding GHG externality measured in units of the consumption good. It remains to show that C_t^* can be recovered under the optimal tax. This can be shown using the representative household's problem as follows. Since we have established a one-to-one relationship between E_t and τ_t, we assume without loss of generality that the planner chooses E_t. Further, assume that E_t is chosen as a function of S_t only.[11] Given $E = E(S)$, k, K, and S, a representative household solves:

$$
\begin{aligned}
V(k,K,S) &= \max_{c,\tilde{k}'}\min_{\hat{\pi}(\gamma)}\left\{u(c)+\beta\hat{E}_\gamma\left[V(k',K',S')+\alpha\log\left(\frac{\hat{\pi}(\gamma)}{\pi(\gamma)}\right)\right]\right\} && (54)\\
& \quad s.t. \\
c+\tilde{k}' &= r(K,S)k+\tau(K,S)E(S)+\pi^{profit} && (55)\\
\tilde{K}' &= G(K,S) && (56)\\
k' &= e^{-\gamma S'}\tilde{k}' && (57)\\
K' &= e^{-\gamma S'}\tilde{K}' && (58)\\
S' &= S+\phi_0 E(S) && (59)
\end{aligned}
$$

where $u(c) = \log(c)$, $r(K,S) = \theta K^{\theta-1}[E(S)]^\nu$, $\tau(K,S) = \nu K^\theta[E(S)]^{\nu-1}$, π^{profit} is the firm's profit, and $\tilde{K}' = G(K,S)$ is the equilibrium transition law for the aggregate capital stock. Here, (k,K,S) stands for the beginning-of-period and $(\tilde{k}',\tilde{K}',S')$ for the end-of-period state, respectively. Notice that $(\tilde{k}',\tilde{K}')$ is not equal to the beginning-of-next-period state, (k',K'), due to capital deterioration by a factor $e^{-\gamma S'}$.

In addition, $\hat{E}_\gamma$ is calculated with respect to the worst case distribution of γ, $\hat{\pi}(\gamma)$, as chosen by the minimizing player. Since the minimizing player moves after the maximizing player, the worst distribution is, in general, conditional on the end-of-period state, $(\tilde{k}',\tilde{K}',S')$. It can be shown that the optimal consumption sequence satisfies the following Euler equation:

$$
u'(c^*) = \beta\frac{\int e^{-\gamma S'}r(K',S')u'(c'^*)e^{-\frac{V(k',K',S')}{\alpha}}\pi(\gamma)d\gamma}{\int e^{-\frac{V(k',K',S')}{\alpha}}\pi(\gamma)d\gamma} \qquad (60)
$$

This yields the following Proposition.

Proposition 2. *If (A1) - (A10) are satisfied, the planner sets* $E = c_E(1-\Delta S)$, *or equivalently* $\tau_t = \frac{\lambda^s}{u'(C^*)}$. *The tax proceeds are rebated lump-sum to the representative consumer. The competitive equilibrium allocation coincides with the solution to the planner's problem. That is,* $c^* = C^* = (1-\beta\theta)K^\theta[c_E(1-\Delta S)]^\nu$.

[11] This is without loss of generality, since our goal is to recover the optimal emissions in eq(47), which only depends on S_t.

4 The Computational Solution and Calibration

In this section, we first extend the analytical model by relaxing assumptions **(A6.1)** and **(A6.2)**. For our baseline model, we will assume that the approximating distribution of γ, $\pi(\gamma)$ is exponential. With $\phi_L > 0$, we need to introduce two additional state variables, P and T, to replace S, which is the sum of P and T. We will also relax **(A7)** by incorporating a "coal" and a "green" sector into the model. Furthermore, we will relax **(A8)** and **(A9)** by allowing A_2N_2 and A_3N_3 to grow at a rate of two percent per year. Last, we will drop **(A10)**.

The social planner's problem becomes:

$$V(K,N,P,T,R) = \max_{\{C,E_1,E_2,E_3,E,\tilde{K}',P',T',S',R'\}} \min_{m(\gamma)}$$

$$\{u(C) + \beta \int [m(\gamma)V(K',N',P',T',R') + \alpha m(\gamma)\log m(\gamma)]\, \pi(\gamma)\mathrm{d}\gamma\} \quad (61)$$

$s.t.$

$$E = (\kappa_1 E_1^\rho + \kappa_2 E_2^\rho + \kappa_3 E_3^\rho)^{1/\rho} \quad (62)$$

$$\tilde{K}' = F\left(K, N(1 - \frac{E_2}{A_2N} - \frac{E_3}{A_3N}), E\right) - C \quad (63)$$

$$K' = h(S',\gamma)\tilde{K}' \quad (64)$$

$$A_2'N' = (1+g)A_2N \quad (65)$$

$$A_3'N' = (1+g)A_3N \quad (66)$$

$$R' = R - E_1 \geq 0 \quad (67)$$

$$P' = P + \phi_L(E_1 + E_2) \quad (68)$$

$$T' = (1-\phi)T + (1-\phi_L)\phi_0(E_1+E_2) \quad (69)$$

$$S' = P' + T' \quad (70)$$

$$1 = \int m(\gamma)\pi(\gamma)\mathrm{d}\gamma \quad (71)$$

To solve this problem, first note that most of the analysis conducted in Section 3 carries over. The only difference is that the function $f(\cdot)$ no longer has a closed form expression. We apply the outer-inner loop method used in Section 3. The inner loop minimization problem is unchanged, while the outer loop maximization problem can be solved in parts. For the latter, it is important to note that solving the optimization problem for E_i, P', T', and R' can be carried out separately from solving for C and $\tilde{K}'$ and the solution to the second optimization problem remains the same as in section 3; i.e., $C^* = (1-\beta\theta)Y^*$ and $\tilde{K}'^* = \beta\theta Y^*$, where Y^* denotes the optimal output level. After substituting for C^*, the optimization problem for

E_i, P', T', and R' can be simplified to give the dynamic programming problem below:

$$
\begin{aligned}
f(N,P,T,R) &= \max_{E_1,E_2,E_3,E,P',T',S',R'} \\
&\left\{\frac{1}{1-\beta\theta}\log[(1-\frac{E_2}{A_2N}-\frac{E_3}{A_3N})^{1-\theta-\nu}E^{\nu}]+\beta[f(N',P',T',R')+\alpha\log(1-\Delta S')]\right\} \quad (72)
\end{aligned}
$$

$s.t.$

$$E = (\kappa_1 E_1^{\rho}+\kappa_2 E_2^{\rho}+\kappa_3 E_3^{\rho})^{1/\rho} \quad (73)$$

$$N' = (1+g)N \quad (74)$$

$$R' = R - E_1 \geq 0 \quad (75)$$

$$P' = P + \phi_L(E_1+E_2) \quad (76)$$

$$T' = (1-\phi)T + (1-\phi_L)\phi_0(E_1+E_2) \quad (77)$$

$$S' = P' + T' \quad (78)$$

We now characterize the optimality conditions for E_3, E_2, and E_1, respectively. The first-order condition for E_3 implies[12]

$$\frac{\nu\kappa_3}{E_3^{1-\rho}E^{\rho}} = \frac{1-\theta-\nu}{A_3N_0} \quad (79)$$

The first-order condition for E_2 gives

$$
\begin{aligned}
&\frac{1-\theta-\nu}{A_2N_0} \\
&= \frac{\nu\kappa_2}{E_2^{1-\rho}E^{\rho}} + (1-\beta\theta)\beta\left[\phi_L\left(\frac{\partial f}{\partial P'}-\frac{\alpha\Delta}{1-\Delta S'}\right)+(1-\phi_L)\phi_0\left(\frac{\partial f}{\partial T'}-\frac{\alpha\Delta}{1-\Delta S'}\right)\right] \quad (80)
\end{aligned}
$$

Applying the envelope theorem to P and T gives

$$\frac{\partial f}{\partial P} = \beta\left(\frac{\partial f}{\partial P'}-\frac{\alpha\Delta}{1-\Delta S'}\right) \quad (81)$$

$$\frac{\partial f}{\partial T} = \beta(1-\phi)\left(\frac{\partial f}{\partial T'}-\frac{\alpha\Delta}{1-\Delta S'}\right) \quad (82)$$

Denoting by $\hat{\Lambda}^P = -(1-\beta\theta)\frac{\partial f}{\partial P}$ and by $\hat{\Lambda}^T = -(1-\beta\theta)\frac{\partial f}{\partial T}$ the marginal values of the externality caused by P and T, respectively, the first-order condition for E_2 becomes

$$\frac{1-\theta-\nu}{A_2N_0} = \frac{\nu\kappa_2}{E_2^{1-\rho}E^{\rho}} - \left[\phi_L\hat{\Lambda}^P + \frac{(1-\phi_L)\phi_0}{1-\phi}\hat{\Lambda}^T\right] \quad (83)$$

It is easy to see that the marginal externality caused by E_2 (or E_1) is given by

$$\hat{\Lambda}^S = \phi_L\hat{\Lambda}^P + \frac{(1-\phi_L)\phi_0}{1-\phi}\hat{\Lambda}^T \quad (84)$$

[12] This is the same as equation (24) in GHKT (2012), as the “green" sector does not emit GHG.

Thus, we obtain

$$\frac{\nu\kappa_2}{E_2^{1-\rho}E^{\rho}} - \hat{\Lambda}^S = \frac{1-\theta-\nu}{A_2 N_0} \tag{85}$$

This has the same form as equation (23) in GHKT (2012), but under a different interpretation of $\hat{\Lambda}^S$. To see the difference, it is convenient to restore the time index, t. From eq(81) and eq(82) we have

$$\hat{\Lambda}_t^P = (1-\beta\theta)\alpha\Delta\sum_{j=1}^{+\infty}\frac{\beta^j}{1-\Delta S_{t+j}} = \theta\bar{\gamma}\sum_{j=1}^{+\infty}\frac{\beta^j}{1-\Delta S_{t+j}} \tag{86}$$

$$\hat{\Lambda}_t^T = (1-\beta\theta)\alpha\Delta\sum_{j=1}^{+\infty}\frac{[\beta(1-\phi)]^j}{1-\Delta S_{t+j}} = \theta\bar{\gamma}\sum_{j=1}^{+\infty}\frac{[\beta(1-\phi)]^j}{1-\Delta S_{t+j}} \tag{87}$$

The second equality in either equation is obtained by using $(1-\beta\theta)\alpha\Delta = (1-\beta\theta)\alpha\frac{\bar{A}}{\alpha\lambda} = \theta\lambda^{-1} = \theta\bar{\gamma}$, where $\lambda^{-1} = \bar{\gamma}$ is the mean of γ under the approximating model. It follows immediately that $\hat{\Lambda}_t^S$ can be expressed as

$$\hat{\Lambda}_t^S = \theta\bar{\gamma}\sum_{j=1}^{+\infty}\left[\phi_L\frac{\beta^j}{1-\Delta S_{t+j}} + \frac{(1-\phi_L)\phi_0}{1-\phi}\frac{[\beta(1-\phi)]^j}{1-\Delta S_{t+j}}\right] \tag{88}$$

It is instructive to consider the case when $\alpha\to+\infty$; i.e., when there is no concern about model uncertainty. It is key to realize that $\Delta\to 0$ as $\alpha\to+\infty$. Therefore,

$$\begin{aligned}\lim_{\alpha\to+\infty}\hat{\Lambda}_t^S &= \theta\bar{\gamma}\sum_{j=1}^{+\infty}\left[\phi_L\beta^j + \frac{(1-\phi_L)\phi_0}{1-\phi}[\beta(1-\phi)]^j\right] \\ &= \theta\bar{\gamma}\left[\frac{\phi_L\beta}{1-\beta} + \frac{(1-\phi_L)\phi_0\beta}{1-(1-\phi)\beta}\right] \end{aligned} \tag{89}$$

Contrasting this with equation (12) in GHKT (2012), which gives $\hat{\Lambda}_t^S = \theta\bar{\gamma}\left[\frac{\phi_L}{1-\beta} + \frac{(1-\phi_L)\phi_0}{1-(1-\phi)\beta}\right]$, we can identify two differences. First, here there is an additional term (θ) in eq(89). This is because GHG affect aggregate capital instead of output in our model. Second, the externality related to P and T is weighted by β in eq(89). This is because GHG in our model affect the next period's capital rather than the current one. Finally, it is easy to show that the first-order condition for E_1 yields

$$\frac{\nu\kappa_1}{E_1^{1-\rho}E^{\rho}} - \hat{\Lambda}^S = \beta\left[\frac{\nu\kappa_1}{(E_1')^{1-\rho}(E')^{\rho}} - (\hat{\Lambda}^S)'\right] \tag{90}$$

Note that the operator $\mathbb{E}_t$ does not appear in the right-hand-side, as the planner optimizes under the worst case scenario, rather than averaging over all possible cases.

As the planner's problem in this section has a similar structure as the analytical model, it can be shown in a similar way that an analogues of Propositions 1 and 2 hold in this environment. We numerically solve the above problem for the case where $\alpha = 0.01$ and $\alpha = 100$. We use the same parameter values as in GHKT (2012), except for R_0, which is set to 800 as in Rogner (1997). Figures 4 through Figure 6 plot the computed optimal paths.

Parameter	ϕ	ϕ_L	ϕ_0	θ	ν	β	ρ	$1+g$
Value	0.0228	0.2	0.393	0.3	0.04	0.985^{10}	-0.058	1.02^{10}
Parameter	P_0	T_0	R_0	κ_1	κ_2	$A_{2,0}$	$A_{3,0}$	λ^{-1}
Value	103	699	800	0.5008	0.08916	$7,693$	$1,311$	2.379×10^{-5}

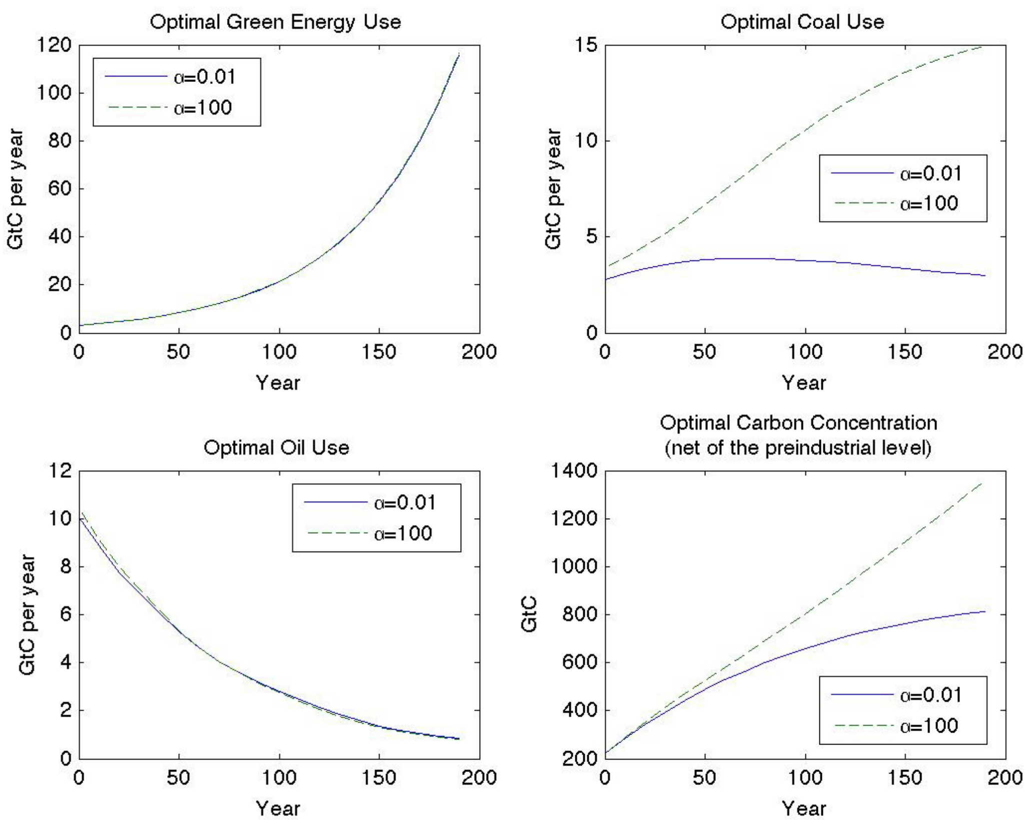

Figure 4: Optimal Use of Energy

Figure 4 describes the optimal paths for green energy, coal, and oil use, as well as the resulting carbon concentration in the atmosphere conditional on different levels of concern about model uncertainty. For simplicity, we refer to the optimal path under $\alpha = 100$ as the "non-robust optimal path," and to the path under $\alpha = 0.01$ as the "robust optimal path." Since the green energy sector does not inject carbon into the atmosphere, the optimal path for the use of green energy does not directly depend on the levels of concern about model uncertainty regarding the externality of carbon emissions. However, since green energy, coal, and oil are substitutes, model uncertainty does affect the use of green energy indirectly, through its impact on the "dirty" energy sectors — coal and oil.

We also find that an increase in the concern about model uncertainty causes a significant decline in the use of coal, while the use of oil is delayed, but only slightly. Note that the supply of oil is finite; thereby the decline rate of oil-use depends not only on the concern about model uncertainty, but also on resource scarcity. As we will show in the next section, an initial stock of oil equaling $R_0 = 800GtC$ is low enough so that the effect of resource scarcity overwhelms the effect of model uncertainty in determining the optimal use of oil. This explains why we do not observe a sharp decrease in the optimal use of oil when the concern about model uncertainty increases. Finally, straightforward calculation shows that the difference between these two optimal paths regarding energy use leads to a significant gap between the associated carbon accumulation paths. Our model predicts that if there is little concern about model uncertainty ($\alpha = 100$), or if our concern about model uncertainty is not incorporated in the model ($\alpha = 0.01$), atmospheric carbon concentrations will reach a level as high as $1350GtC$ (net of the preindustrial level) after 180 years. However, this number can be reduced to about $800GtC$ if the concern about model uncertainty is incorporated and addressed through implementing the corresponding optimal tax in order to restore the optimal energy path when $\alpha = 0.01$.

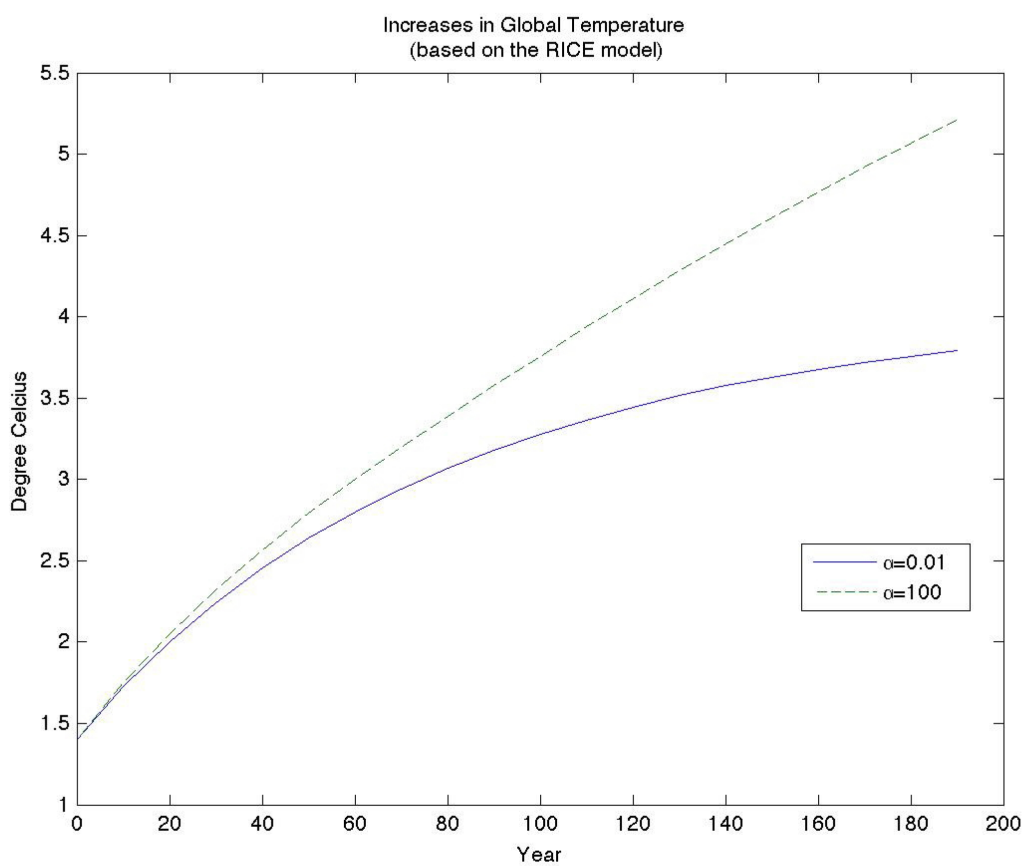

Figure 5: Increase in Global Temperatures

Figure 5 demonstrates a direct consequence of the above results: based on a simple mapping from carbon concentrations to global temperatures used in RICE, $T(S_t) = 3\ln(\frac{S_t}{\bar{S}})/\ln 2$, the global average temperature will rise by 3.8 degree Celsius 180 years from now if the concern about model uncertainty is addressed, and by 5.3 degree Celsius otherwise.

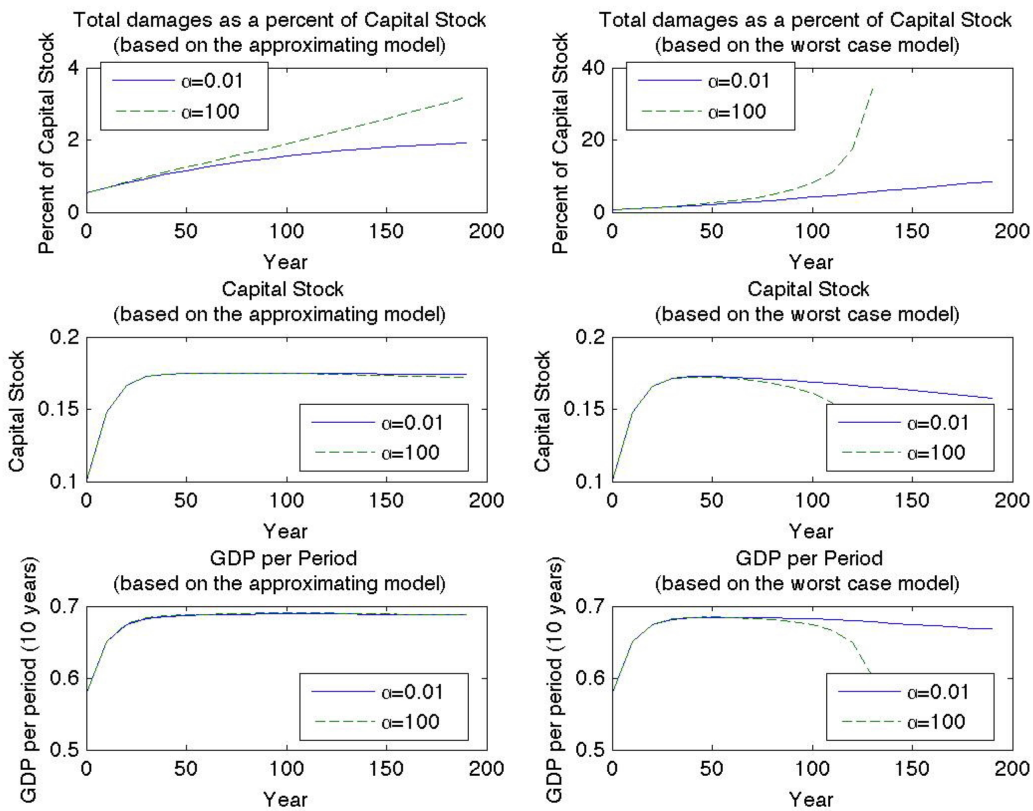

Figure 6: Capital Stock and Output

The graphs in the first (second) column in Figure 6 describe the paths of total damages as a percentage of capital, capital stock, and output, respectively, assuming that the approximating model (worst case model) for γ is the true model.[13] In each graph, the green-dashed line (blue-solid line) represents the outcome when energy is extracted based on the non-robust (robust) optimal path. The main findings can be summarized as follows. If the approximating model for γ is the true model, pursuing the robust optimal path for energy consumption would further reduce total damages by an additional 1 percent 180 years from now. However, due to its more conservative use of oil and coal in the final good sector, such a policy will

[13] To get smooth paths, γ is set to be the expected mean of the approximating (worst case) distritution(s) each period.

also reduce both capital stock and output in the long run. Since utility depends only on consumption (which is proportional to output), this implies that the loss in social welfare due to an overestimated concern about model uncertainty is almost negligible. In contrast, if the true distribution of γ evolves according to the worst case model in each period (second column of Fig. 6), the cost of implementing the non-robust optimal policy is rather large. In fact, the non-robust policy, which overlooks concerns about model uncertainty, will destroy almost the entire capital stock in 120 years, resulting in a big reduction in output and social welfare.[14]

4.1 Varying the Approximating Distribution

Here we explore the consequences of relaxing assumption **(A5)**. We now assume that the approximating distribution of γ is normal with mean $\bar{\gamma}$ and variance σ^2; i.e., $\pi(\gamma) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(\gamma-\bar{\gamma})^2}{2\sigma^2}}$. The switch from exponential to normal distribution creates two key differences. First, the normal distribution provides us with two degrees of freedom: the mean, $\bar{\gamma}$, reflecting the planner's prior expectation regarding damages, and the variance, σ^2, indicating the prior regarding model uncertainty. In comparison, the exponential distribution only used one parameter, λ, which determined both the mean and the variance of γ. Second, as we will see below, assuming that γ is normally distributed can eliminate the "breaking point" for S, which is always present when γ follows an exponential.[15] Thus, under a normal distribution we can accommodate any level of concern about model uncertainty by choosing an arbitrarily small α.

It follows that

$$H(S';\alpha,\bar{A}) = -(\bar{\gamma} + \frac{\bar{A}\sigma^2}{2\alpha}S')\bar{A}S' \tag{91}$$

$$\hat{\pi}^*(\gamma) \sim \mathcal{N}(\bar{\gamma} + \frac{\bar{A}\sigma^2}{\alpha}S'^2, \sigma^2) \tag{92}$$

It is straightforward to show that $H(\cdot)$ is strictly negative, strictly increasing in α, and strictly decreasing in both $\bar{\gamma}$ and σ^2. In addition, the worst case distribution for γ also follows a normal distribution, and $\hat{\pi}^*(\gamma)$ and $\pi(\gamma)$ differ only in their means. That is, when choosing the worst case model, nature only alters the mean of γ, rather than its variance. As a by-product, the relative entropy of $\hat{\pi}^*(\gamma)$ with respect to $\pi^*(\gamma)$ is given by

$$\rho(\hat{\pi}^*(\gamma), \pi^*(\gamma)) = \frac{\bar{A}^2\sigma^2 S'^2}{2\alpha^2} \tag{93}$$

To complete the model based on $\pi(\gamma) \sim \mathcal{N}(\bar{\gamma}, \sigma^2)$, we simply need to replace $\alpha\log(1-\Delta S')$ in eq(72) by $-(\bar{\gamma} + \frac{\bar{A}\sigma^2}{2\alpha}S')\bar{A}S'$. Accordingly, the optimality conditions for E_1, E_2, and E_3

[14] The extreme negative effect on capital, output, and social welfare is partly due to the assumption that the approximating distribution of γ is exponential. As we will show later, the losses are somewhat reduced, though still large, if the approximating distribution of γ is assumed to be normal.

[15] This is because the exponential distribution has a fatter tail than the normal distribution, thus, allowing more room for nature to create a worst-case-scenario for the economy given a level of penalty, α.

remain the same, expect that the values of the externality associated with P, T, and E_2 (or E_1), respectively, are now as follows:

$$\hat{\Lambda}_t^P = \frac{\beta\theta\bar{\gamma}}{1-\beta} + \frac{\theta\bar{A}\sigma^2}{\alpha}\sum_{j=1}^{+\infty}\beta^j S_{t+j} \tag{94}$$

$$\hat{\Lambda}_t^T = \frac{\beta(1-\phi)\theta\bar{\gamma}}{1-\beta(1-\phi)} + \frac{\theta\bar{A}\sigma^2}{\alpha}\sum_{j=1}^{+\infty}[\beta(1-\phi)]^j S_{t+j} \tag{95}$$

$$\hat{\Lambda}_t^S = \phi_L\hat{\Lambda}_t^P + \frac{(1-\phi_L)\phi_0}{1-\phi}\hat{\Lambda}_t^T \tag{96}$$

Note that $\hat{\Lambda}_t^S$ reduces to the previous expression as $\alpha \to +\infty$, or as $\sigma^2 \to 0$. That is,

$$\hat{\Lambda}_t^S = \theta\bar{\gamma}\left[\frac{\phi_L\beta}{1-\beta} + \frac{(1-\phi_L)\phi_0\beta}{1-(1-\phi)\beta}\right] \text{ as } \alpha \to +\infty, \text{ or } \sigma^2 \to 0$$

We will consider three cases regarding the initial stock of fossil fuel: $R_0 = 253.8$, $R_0 = 8000$, and $R_0 = \infty$.[16] For each case, we numerically solve the above problem for $\alpha = 0.01$ and for $\alpha = +\infty$. To draw an even closer comparison with GHKT (2012), we have re-scaled γ by a factor of $1/\theta$, where θ is the share of capital. The reason is that, given a Cobb-Douglas specification in final goods production, and given 100% depreciation of capital, a proportional damage $e^{-\gamma S'}$ on capital is equivalent to a proportional damage $e^{-\theta\gamma S'}$ on output. Accordingly, the mean and variance of γ in the approximating model are set to $\bar{\gamma} = 7.93 \times 10^{-5}$ and $\sigma^2 = 2.65 \times 10^{-8}$, respectively.

Below we plot the same quantities as those shown in Fig. 4 through Fig. 6, but under the assumption that the approximating distribution of γ is normal. Here, our attention will be paid primarily in comparing the effects of model uncertainty on optimal oil use under different values of R_0. As what we have discussed earlier, holding other parameters fixed, the optimal path of oil consumption is determined jointly by the resource scarcity effect and the model uncertainty effect. We can hardly see a significant difference between the robust and non-robust optimal path of oil consumption when the scarcity effect dominates (i.e., when R_0 is sufficiently small). To emphasize the impact of model uncertainty, we will consider different values of R_0. Figure 7 shows that, when $R_0 = 253.8GtC$, the non-robust optimal paths are able to replicate their counterparts in GHKT (2012). The concern about model uncertainty delays the optimal use of oil only slightly. Fig. 10 displays a different pattern. When R_0 is set to $8000GtC$, although both paths are still decreasing over time, the concern about model uncertainty does discourage the use of oil substantively. Finally, as R_0 goes to infinity, as shown in Fig. 12, we observe a qualitative difference between the two paths. On the one hand, the non-robust optimal path allows the use of oil to grow unboundedly, partially due to the technology progress in the coal and green sectors. On the other hand,

[16] Note that the total stock of oil and gas is estimated to be over $15,000GtC$ if methane hydrates are included.

the increasing trend of oil consumption is curbed due to the concern about the externality caused by carbon emissions.

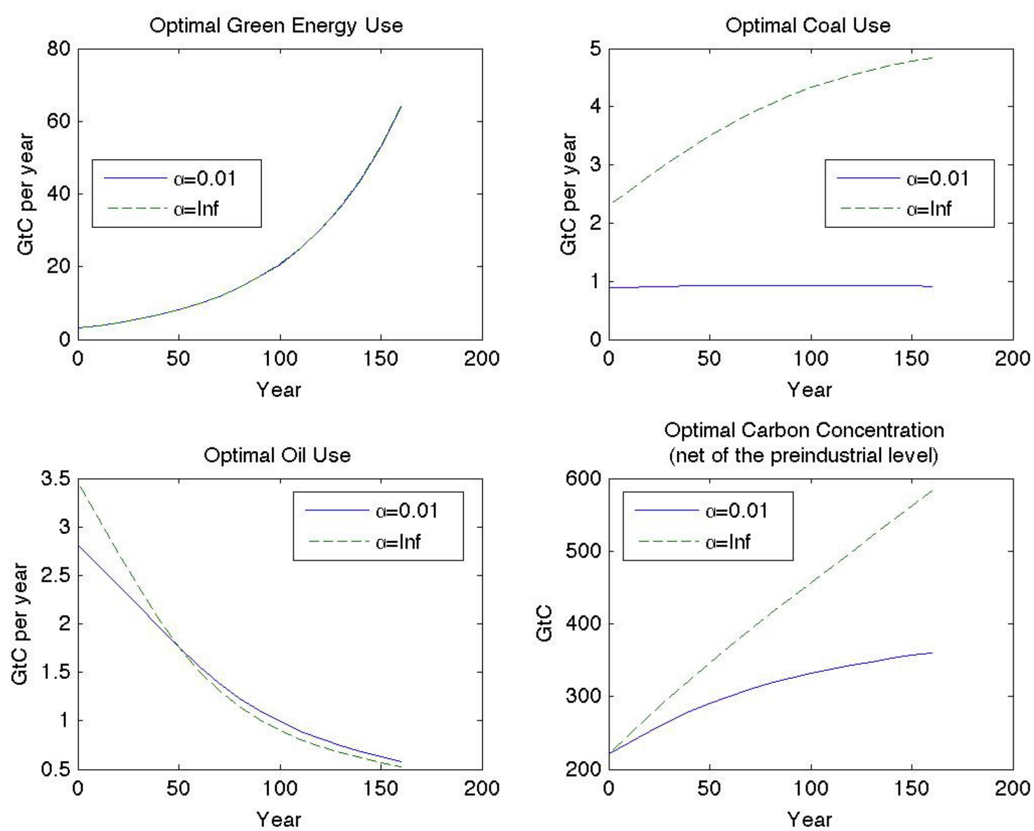

Figure 7: Optimal Use of Energy when $R_0 = 253.8$

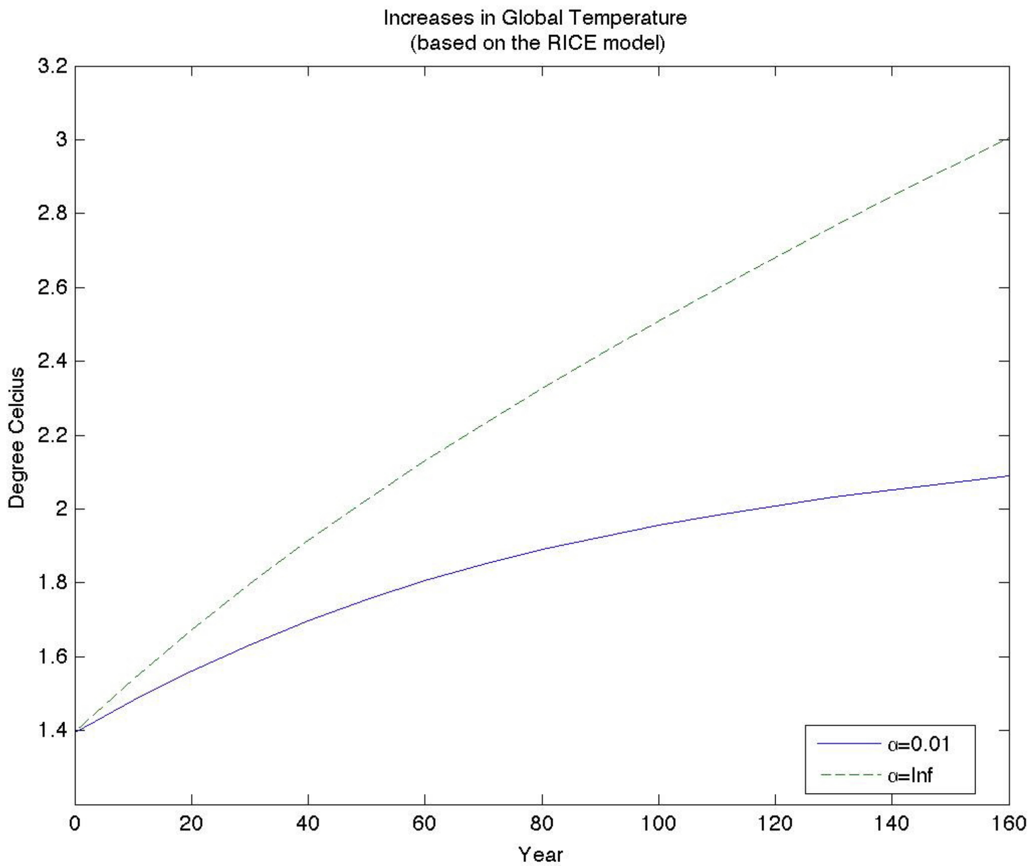

Figure 8: Increase in Global Temperatures when $R_0 = 253.8$

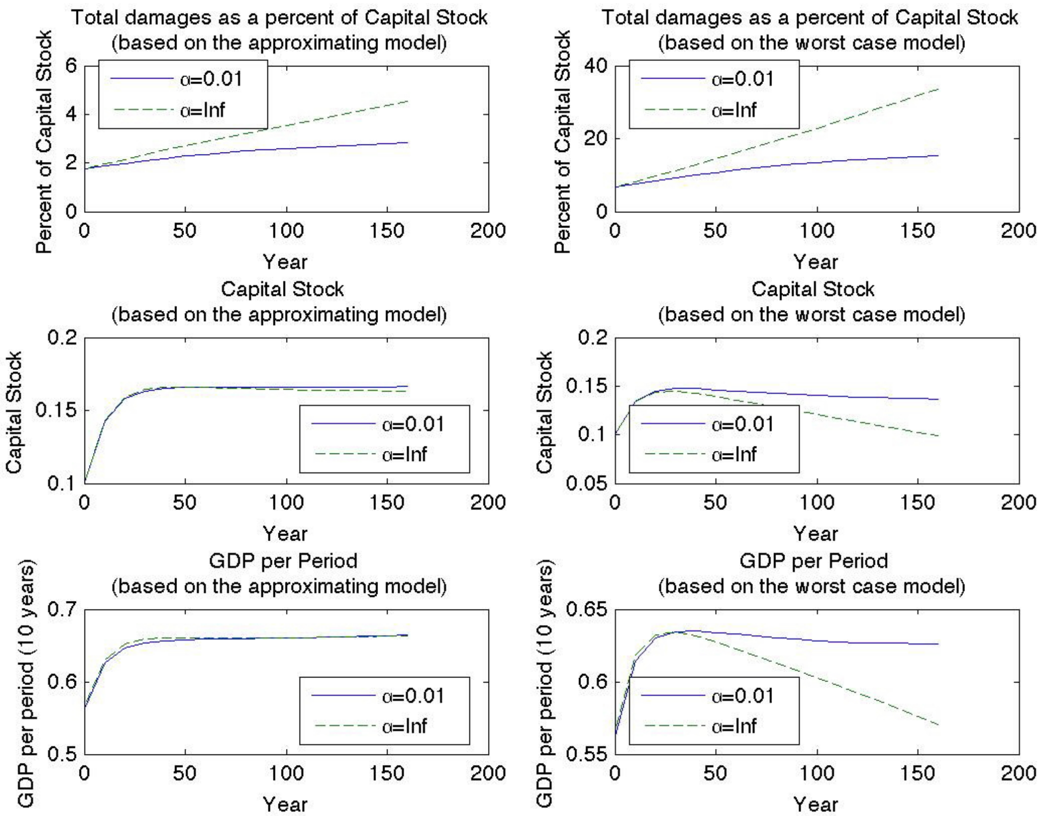

Figure 9: Capital Stock and Output when $R_0 = 253.8$

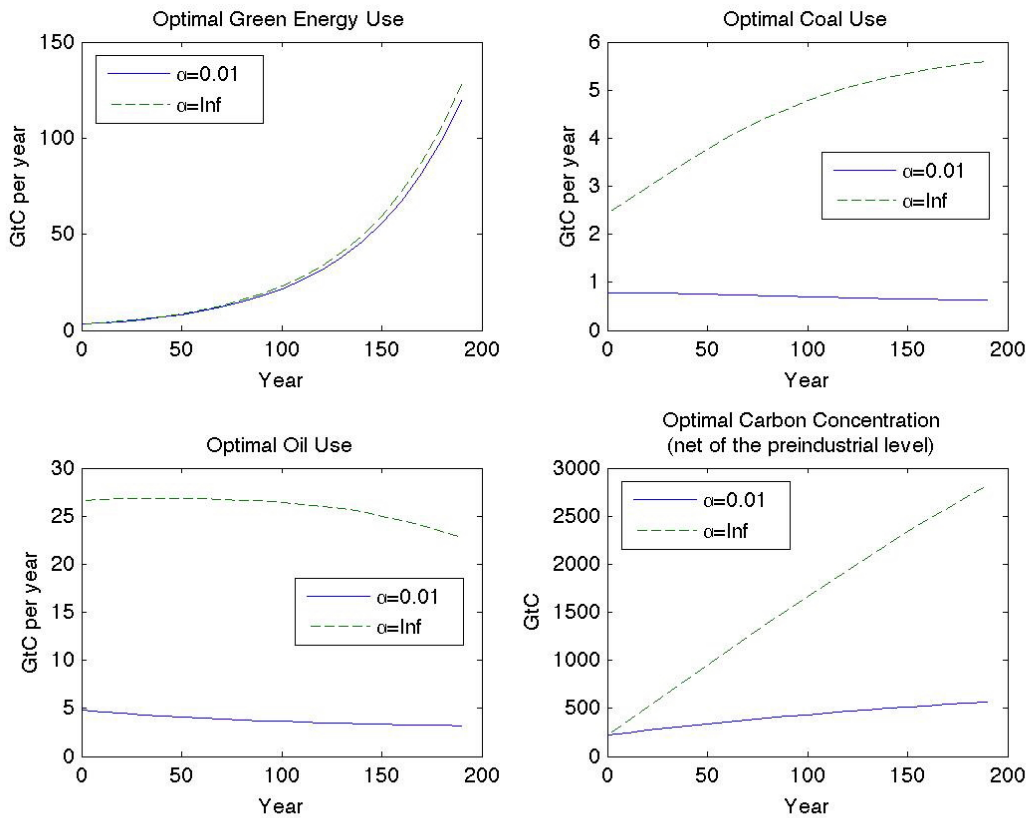

Figure 10: Optimal Use of Energy when $R_0 = 8000$

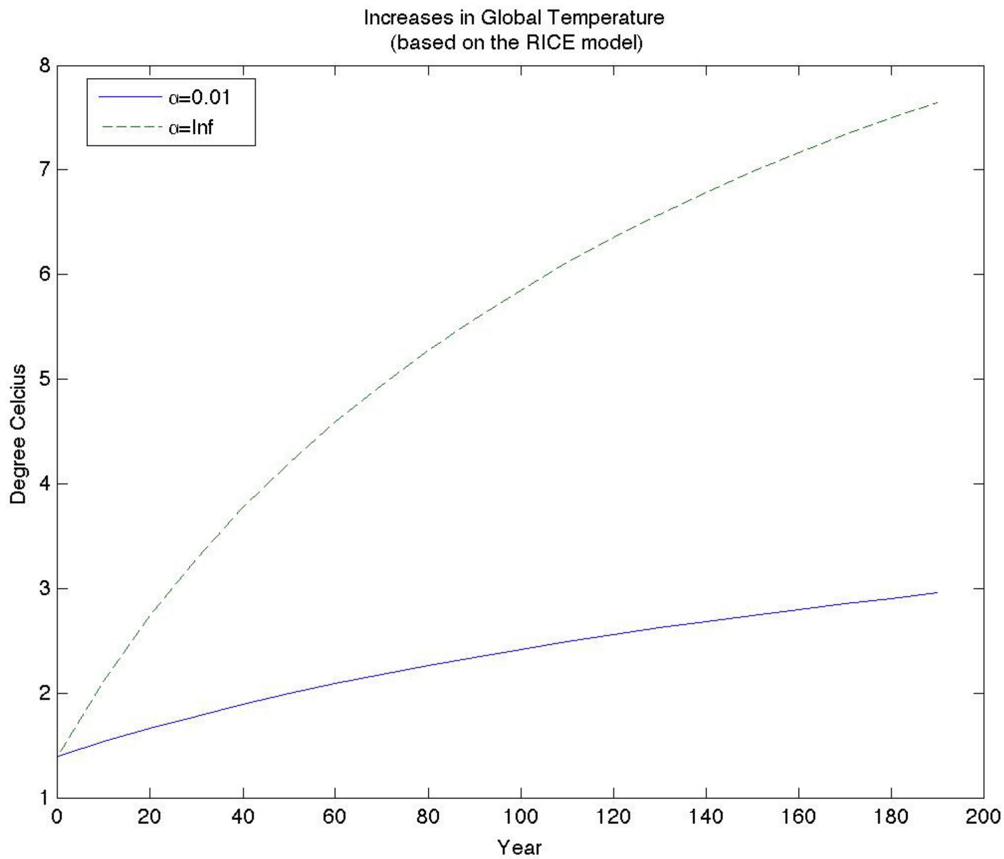

Figure 11: Increase in Global Temperatures when $R_0 = 8000$

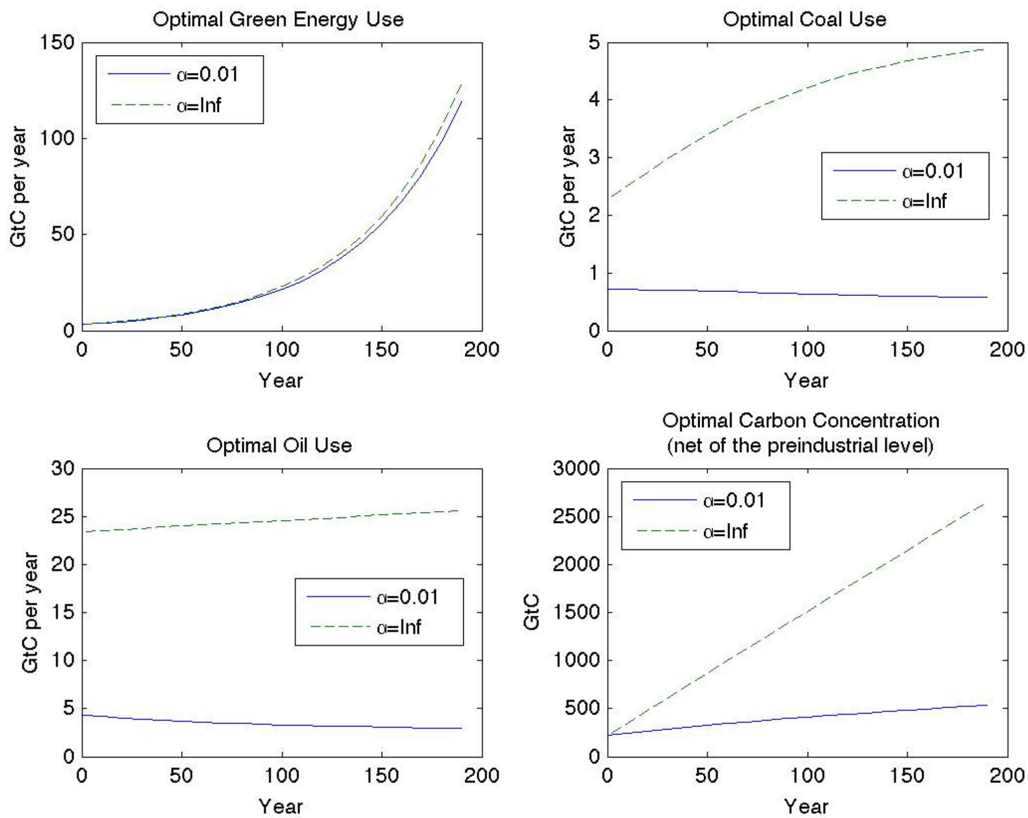

Figure 12: Optimal Use of Energy when $R_0 = \infty$

GHKT (2012) assume $R_o = 253.8GtC$ and estimate damages of *\$56.9/ton of carbon* using an annual discount rate of *1.5%* and *\$496/ton* under a rate of *0.1%*. As opposed to their results, our model assumes that carbon emissions in the current period affect the next period capital stock and output, instead of current ones. It implies that when $\beta = 0.985^{10}$ and there is no concern of robustness (i.e., $\alpha = \infty$), the welfare loss given by our model equals $0.985^{10} \times 56.4 = \$48.5/ton$. This number is independent of the approximating distribution for γ, the initial stock of oil, and the future path of carbon concentration. When $\alpha = 0.01$, these factors matter. If the approximating distribution is normal, the losses are given as follows.

R_o/α	0.01	0.1	1	100	∞
253.8 GtC	239.60	70.65	50.85	48.52	48.49
8000 GtC	276.60	90.60	55.08	48.57	48.49
∞ GtC	318.70	103.06	63.42	56.49	48.49

(97)

4.2 Varying the Resource Feasibility Constraint (with Constrained Coal and infinite oil)

Here, we report the results for the case where oil is in infinite supply, while coal is constrained with an initial stock $R_{coal} = 666GtC$. This case demonstrates that the optimal use of oil mimics that when both oil and coal are in infinite supply. In addition, the use of coal increases steadily at the beginning and then starts to drop.

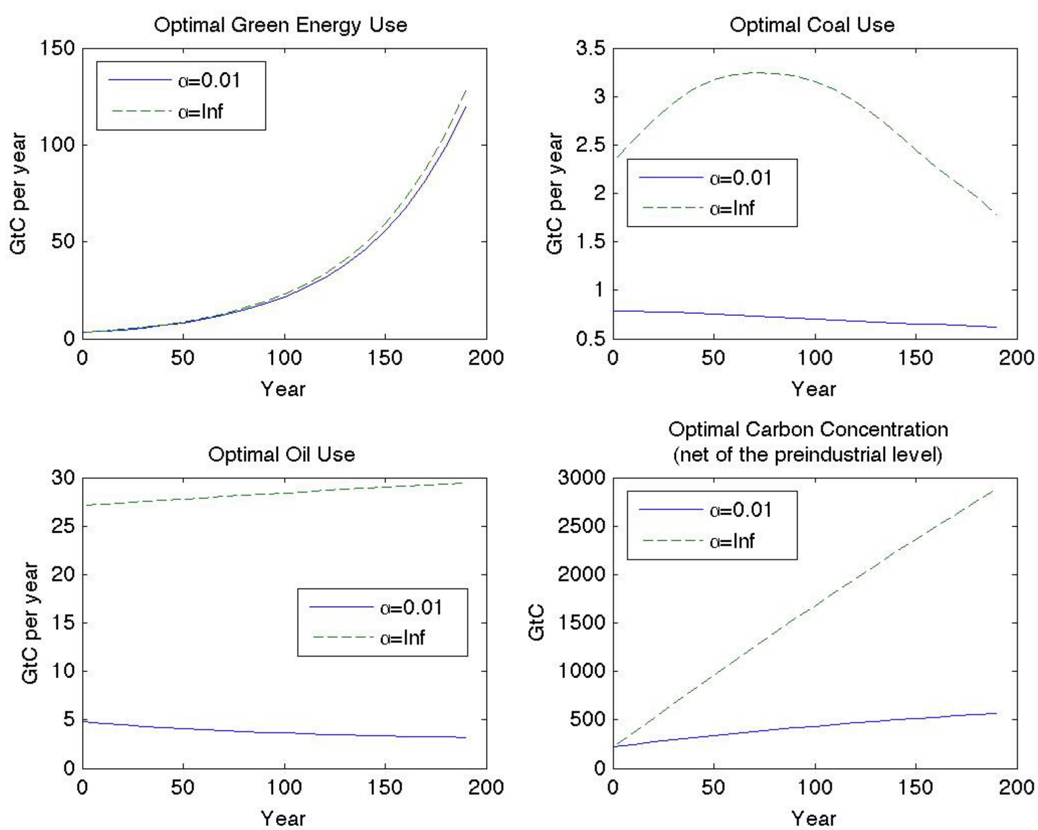

Figure 13: Optimal Use of Energy when $R_{coal} = 666$

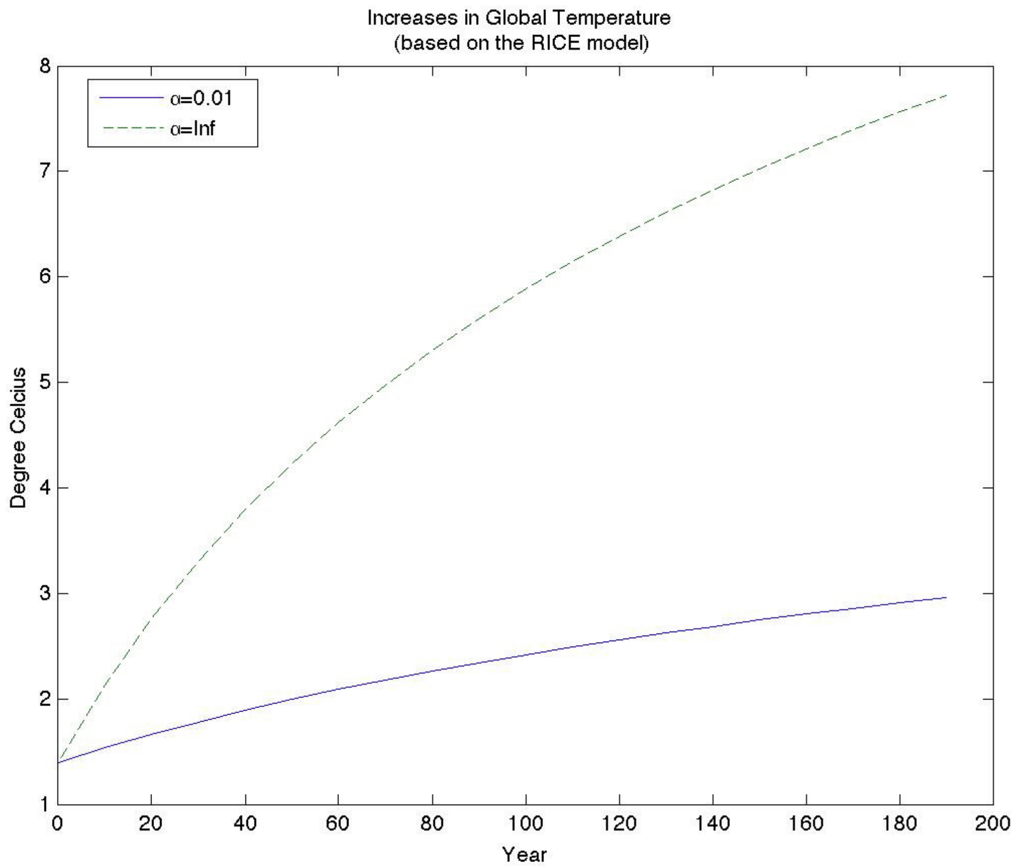

Figure 14: Increase in Global Temperature when $R_{coal} = 666$

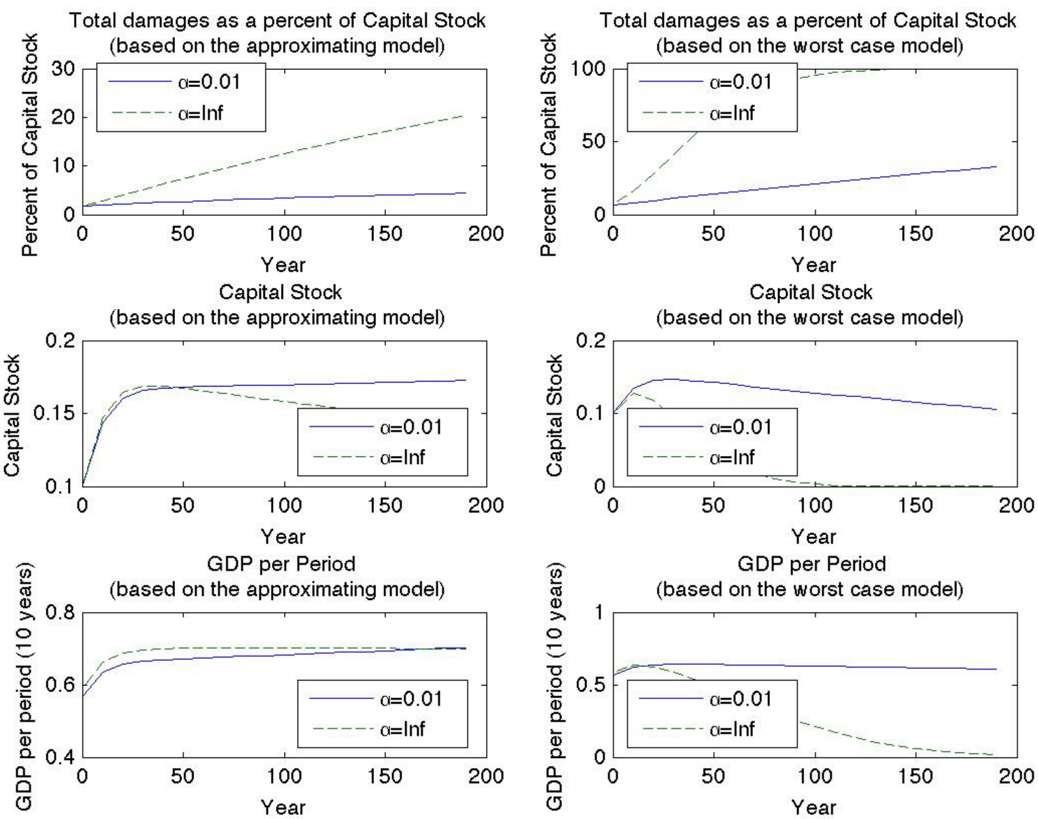

Figure 15: Capital Stock and Output when $R_{coal} = 666$

5 Conclusion

We studied optimal taxation in a dynamic stochastic general equilibrium model where agents are concerned about model uncertainty regarding climate change. We obtained a sharp analytical solution for the implied externality and we characterize dynamic optimal taxation. A small increase in the concern about model uncertainty can cause a significant drop in optimal energy extraction. The optimal tax which restores the social optimal allocation is Pigouvian. Under more general assumptions, we developed a recursive method that allows us to solve the model computationally. The introduction of uncertainty matters in a number of ways, both qualitatively and quantitatively.

Our model can be extended in many ways. In the current version, the growth rate of renewables is assumed to be independent from the concern about model uncertainty. It would be interesting to endogenize growth in renewable energy productivity. A related extension could involve using a distortionary tax on labor to subsidize R&D in renewables in order to study the effects on energy composition and growth. Additionally, we could study a version where coal supply is a state variable, while assuming infinite gas and oil supply.

6 Appendix

We demonstrate that the optimal level of GHG, E^*, has the following properties: $\frac{\partial E^*}{\partial \delta} < 0$ and $\frac{\partial E^*}{\partial \delta}|_{\delta=0} = -\infty$, where δ is the upper bound for entropy allowed in the constraint game.

Proof. Recall that $E^* = c_E(1-\Delta S)$ and $\delta = \log(1-\Delta S'^*)+\frac{\Delta S'^*}{1-\Delta S'^*}$, where $S'^* = S+\phi_0 c_E(1-\Delta S)$. Define $a = \alpha^{-1}$ and $b = 1-\Delta S'^* = (1-\Delta\phi_0 c_E)(1-\Delta S)$. It follows immediately that E^* is decreasing in a. In addition, since both Δ and c_E are functions of a, it follows that b is a function of a:

$$b(a) = [1-\Delta(a)\phi_0 c_E(a)][1-\Delta(a)S]$$

It is easy to see that b is decreasing in a. Thus, it defines a as an implicit function of b with a negative slope. Moreover, we can rewrite δ as:

$$\delta = \log b + \frac{1-b}{b},$$

which defines b as an implicit function of δ. Direct calculation shows that $\frac{\partial b}{\partial \delta} = -\frac{b^2}{1-b} < 0$, as $b \in (0,1)$. Thus,

$$\frac{\partial E^*}{\partial \delta} = \frac{\partial E^*}{\partial a}\frac{\partial a}{\partial b}\frac{\partial b}{\partial \delta} < 0$$

Evaluating this at $\delta = 0$, we get that

$$\frac{\partial E^*}{\partial \delta}|_{\delta=0} = \left(\frac{\partial E^*}{\partial a}|_{a=0}\right)\left(\frac{\partial a}{\partial b}|_{b=1}\right)\left(\frac{\partial b}{\partial \delta}|_{\delta=0}\right)$$

It is straightforward to show that the first two terms on the right hand side in the above expression are strictly negative and finite, and the last term goes to $-\infty$. Therefore, $\frac{\partial E^*}{\partial \delta}|_{\delta=0} = -\infty$. □

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