

RISE | RICE INITIATIVE *for the* STUDY *of* ECONOMICS

RISE Working Paper 14-001

“Some Unpleasant Bargaining Arithmetic?”

by

Hülya Eraslan, Antonio Merlo



RICE

Department of Economics

Baker Hall, MS22

6100 Main Street, Houston, Texas 77005

<https://economics.rice.edu>

Some Unpleasant Bargaining Arithmetic?¹

Hülya Eraslan²

Antonio Merlo³

August 2014

¹We thank seminar and conference participants at several institutions for their useful comments and suggestions.

²Department of Economics, Rice University, <eraslan@rice.edu>.

³Department of Economics, Rice University, <amerlo@rice.edu>.

Abstract

It is commonly believed that, since unanimity rule safeguards the rights of each individual, it protects minorities from the possibility of expropriation, thus yielding more equitable outcomes than majority rule. We show that this is not necessarily the case in bargaining environments. We study a multilateral bargaining model à la Baron and Ferejohn (1989), where players are heterogeneous with respect to the potential surplus they bring to the bargaining table. We show that unanimity rule may generate equilibrium outcomes that are more unequal (or less equitable) than under majority rule. In fact, as players become perfectly patient, we show that the more inclusive the voting rule, the less equitable the equilibrium allocations.

JEL Classification: C78, D70.

Keywords: Multilateral bargaining, voting rules, inequality.

1 Introduction

Multilateral bargaining is a staple of political economy, as many political negotiations entail bargaining among several players over the allocation of some surplus (e.g., legislative bargaining, government formation, domestic and international policy negotiations). Whenever negotiations involve more than two players, the voting rule that is used to specify how agreements are reached plays a fundamental role in determining the allocations that are ultimately agreed upon. Starting with the seminal contribution of Baron and Ferejohn (1989), several articles have studied the relative performance of alternative voting rules in multilateral bargaining models (e.g., Banks and Duggan (2000), Baron and Kalai (1993), Eraslan (2002), Eraslan and Merlo (2002), Harrington (1990), Yildirim (2007)). The emphasis of those papers, however, has been primarily on efficiency. In this paper, we focus on the distributional consequences (or equity properties) of different voting rules.

It is commonly believed that, since unanimity rule safeguards the rights of each individual, it protects minorities from the possibility of expropriation, thus yielding more equitable outcomes than majority rule (e.g., Buchanan and Tullock (1962) and Acemoglu and Robinson (2012)). We show that this is not necessarily the case in bargaining environments. We study a multilateral bargaining model à la Baron and Ferejohn (1989), where players are heterogeneous with respect to the potential surplus they bring to the bargaining table. We show that unanimity rule may generate equilibrium outcomes that are more unequal (or less equitable) than equilibrium outcomes under majority rule. In fact, as players become perfectly patient, we show that the more inclusive the voting rule with respect to the number of votes required to induce agreement, the less equitable the equilibrium allocations. These results are a direct implication of basic insights from bargaining theory (some *unpleasant bargaining arithmetic?*).

Like in Baron and Ferejohn (1989), we study an infinite-horizon, n -player bargaining model where in every period each player is randomly selected with equal probability to make a proposal on how to allocate some surplus. The players' payoffs are linear in the amount of surplus they receive and they evaluate future payoffs using a common discount factor. The voting rule specifies the minimum number of players who have to vote in favor of a proposal

for it to be implemented. This number varies from one (dictatorship) to n (unanimity). Our point of departure from the canonical model is to assume that players differ with respect to the amount of surplus they have available for distribution in the event they are selected as proposers. This represents the only source of heterogeneity among players, and is taken to be an inalienable characteristic of each individual. For example, different players may have different (innate) abilities in formulating and implementing proposals, which make them inherently more or less “productive” and cannot be transferred across players or mimicked by others.¹ One of the most striking results we obtain is that under the unanimity rule it is possible for the most productive player to appropriate the entire surplus in equilibrium. On the other hand, under any other voting rule it is always the case that other players also have positive equilibrium payoffs.

There is a recent, related literature that studies the implications of alternative voting rules in a variety of economic and political environments. Dixit, Grossman and Gul (2000) analyze the extent to which political compromise arises in a dynamic environment where two parties interact repeatedly and their political strength changes stochastically over time according to a Markov process. The party that is in power at any given time (i.e., the party whose political strength exceeds a given threshold determined by the voting rule), decides to what extent it is willing to share the available surplus with the opposition (i.e., the political compromise). They show that, depending on the degree of persistence in the parties’ political strength, there may be less political compromise, and hence more inequality, under supermajority (where if neither party’s strength exceeds the designated threshold, then both parties must agree to any policy change), than under simple majority rule.

Compte and Jehiel (2010) compare the performance of alternative voting rules in a collective search environment where exogenously specified proposals, which may differ along many dimensions, are drawn independently from a known distribution. A committee considers new proposals sequentially and search stops when the current proposal receives the support of a given number of committee members specified by the voting rule. They show that, depending on the voting rule and on the number of dimensions of the policy space,

¹Baccara and Razin (2007) study a model where firms bargain over the implementation of new ideas and the distribution of the rents they generate. Like in our model, people can only propose their own ideas.

some committee members may have no real voting power, “in the sense that small changes in the objectives or preferences of such members would not affect at all the set of possible agreements” (p. 190).

Finally, Nunnari (2012) studies the consequences of veto power in infinitely repeated legislative bargaining environments where agreement on a policy at any given time determines the new status quo policy for future negotiations. He shows that when agreement on a policy proposal requires a simple majority, but one legislator has the right to block decisions (i.e., the legislator has a veto right), eventually, that legislator appropriates all the surplus.²

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3, we characterize the equilibrium payoffs for different voting rules. In Section 4, we compare the equilibrium implications of different voting rules with respect to their equity properties. We conclude in Section 5.

2 The Model

Consider a situation where $n > 2$ players have to collectively decide how to allocate some surplus. Each player i is endowed with a *potential surplus* $y_i > 0$, denoting the amount of surplus she would have available for distribution if selected as the proposer.³ As mentioned in the Introduction, this situation would arise, for example, in an environment where players are heterogeneous with respect to their ability in formulating and implementing proposals, which makes them inherently more or less “productive” and cannot be transferred across players or mimicked by others (e.g., Baccara and Razin (2007)). We enumerate the players from the least productive to the most productive, i.e., $y_1 \leq y_2 \leq \dots \leq y_n$.

In each period, a player is randomly offered the possibility of submitting a proposal with probability $1/n$.⁴ The selected player i may then make a proposal specifying the way she

²Yildirim (2010) studies the distributional consequences of alternative bargaining protocols. He considers bargaining environments where players can contest the right to make proposals and compares an environment where proposal rights are determined once-and-for-all before the beginning of a negotiation, to one where a contest determining the identity of the proposer takes place prior to each bargaining round. He shows that when agreement requires unanimous consent and players differs with respect to their discount factor, equilibrium allocations are relatively more unequal in the former environment than in the latter.

³Throughout the paper we adopt the convention of using "she" to refer to player i and "he" for any other player.

⁴Since the focus of our analysis is to study the equity properties of alternative voting rules, we deliber-

would distribute surplus y_i among the players, or forego the opportunity and pass. If a proposal is submitted, all players then vote (sequentially) on whether or not to approve it. If at least $q \in \{1, \dots, n\}$ people including the proposer accept the proposal, the game ends and the surplus is shared according to the accepted proposal. Otherwise, a new player is selected as the proposer and the process repeats itself (possibly *ad infinitum*).

Players have an identical, single date, von Neumann-Morgenstern payoff function that is linear in their own share of the surplus, and discount the future at a common discount factor $\delta \in (0, 1)$. In the event that agreement is never reached, all players receive a payoff of zero.

If $q = n$, then the agreement rule is unanimity and the game is a special case of the stochastic bargaining model of Merlo and Wilson (1995, 1998) in which the “cake” process and the “proposer” process are perfectly correlated. If n is odd and $q = (n + 1)/2$, then the agreement rule is majority rule as in Eraslan and Merlo (2002). If, in addition, the surplus available for distribution is the same for all players, i.e., $y_1 = \dots = y_n$, then the game reduces to the one studied by Baron and Ferejohn (1989). For any $q \in \{1, \dots, n\}$, we refer to the voting rule as a q -quota rule.

Let h^t denote the past history of the game up to time t (i.e, the identity of the previous proposers, whether they made proposals, the proposals they made if they made any, and how each player voted for these proposals), together with the identity of the current proposer and the proposal s/he made if s/he made one. A (behavior) strategy for player i , ψ_i , is a probability distribution over feasible actions for each date t and history at date t . A strategy profile ψ is an n -tuple of strategies, one for each player. Let $G(h^t)$ denote the game from date t on with history h^t . Let $\psi|h^t$ denote the restriction of ψ to the histories consistent with h^t . Then $\psi|h^t$ is a strategy profile on $G(h^t)$. A strategy profile ψ is subgame perfect (SP) if, for every history h^t , $\psi|h^t$ is a Nash equilibrium of $G(h^t)$. A strategy profile is stationary if the actions prescribed at any history depend only on the proposer and offer. A stationary, subgame perfect (SSP) outcome and payoff are the outcome and payoff generated by an SSP strategy profile.

It is well known that in multilateral bargaining games like the one considered here there

ately assume an egalitarian bargaining protocol. Eraslan (2002) and Merlo and Wilson (1995, 1998) study bargaining environments where players differ with respect to the probability of being selected as proposer and show that equilibrium payoffs are non-decreasing in such probability, for any voting rule.

is multiplicity of subgame perfect equilibria even under unanimity rule (e.g., Sutton (1986)). However, it has also been recognized that stationarity is typically able to select a unique equilibrium (e.g., Baron and Ferejohn (1989), Merlo and Wilson (1995)). Thus, we restrict attention to SSP equilibria.

3 Characterization of SSP Payoffs

In this section, we characterize the set of SSP payoffs and study their properties. Given an SSP payoff vector v , player i accepts a proposal x if $x_i \geq v_i$ and rejects it if $x_i < v_i$.⁵ Let $r_{ij}(v)$ denote the probability that i includes j in her coalition when she is selected as the proposer and when the payoff vector is v (i.e., the probability that i offers j a payoff equal to v_j), and $r_i(v) = (r_{i1}(v), \dots, r_{in}(v))$. Let $w_i(v)$ denote the *total cost* to player i of her coalition partners (i.e., the total amount of surplus player i has to offer to her coalition partners to induce them to support her proposal), that is,

$$w_i(v) = \sum_{j=1}^n r_{ij}(v)v_j. \quad (1)$$

We maintain the convention that each player i includes herself as a coalition partner (i.e., $r_{ii}(v) = 1$). This is without loss of generality, since the payments made to self cancel out with the distribution received from self.

Since the game we consider is a stochastic bargaining game, agreement need not be reached immediately (Merlo and Wilson (1995, 1998)). In particular, agreement is not reached in any given period if the surplus available in that period is sufficiently small relative to the expected surplus next period. Given a payoff vector v , let $\alpha_i(v) \in [0, 1]$ denote the probability that agreement is reached when i is the proposer. In equilibrium, we must have

$$\alpha_i(v) = \begin{cases} 1 & \text{if } y_i - w_i(v) > 0, \\ 0 & \text{if } y_i - w_i(v) < 0. \end{cases} \quad (2)$$

⁵Although player i is indifferent between accepting and rejecting when $x_i = v_i$, in equilibrium she accepts with probability 1. If not, the proposer does not have an optimal proposal.

If $y_i = w_i(v)$, when selected as the proposer player i is indifferent between passing and proposing and therefore $\alpha_i(v)$ can take any value in $[0, 1]$.

If player i is not the proposer, then she receives her continuation payoff when player j is the proposer in one of two ways: either player j includes player i in his coalition, or player j passes and the bargaining game continues to the next period. Let $\mu_{ji}(v; r(v))$ denote the probability that i receives her continuation payoff v_i when j is the proposer and the payoff vector is v , given the offer probabilities $r(v) = [r_i(v)]_{i=1}^n$. Formally,

$$\mu_{ij}(v; r(v)) = \alpha_i(v)r_{ij}(v) + (1 - \alpha_i(v)). \quad (3)$$

Finally, let

$$\mu_i(v; r(v)) = \sum_{j=1}^n \frac{1}{n} \mu_{ji}(v; r(v)) \quad (4)$$

denote the total probability that i receives her continuation payoff v_i either because of delay or because she is included in the winning coalition.

The following proposition characterizes the SSP payoff vectors for any q -quota game.

Proposition 1 *v is an SSP payoff vector for the q -quota game, $q \in \{1, \dots, n\}$, if and only if for all $i = 1, \dots, n$*

$$v_i = \delta \left[\frac{1}{n} \alpha_i(v) (y_i - w_i(v)) + \mu_i(v; r(v)) v_i \right] \quad (5)$$

and

$$\begin{aligned} r_i(v) \in \arg \min_{z \in [0,1]^n} & \sum_k z_k v_k \\ \text{subject to} & \sum_{k \neq i} z_k = q - 1 \text{ and } z_i = 1. \end{aligned} \quad (6)$$

Proof. Suppose the SSP payoff vector is given by v . Let i denote the proposer, and consider an SSP response to a proposal x by player j . Player j accepts the proposal if $x_j \geq v_j$ and rejects it if $x_j < v_j$. Note that the proposer needs only $q - 1$ votes in addition to her vote for a proposal to be accepted. Then, if the proposer decides to make an offer that will be

accepted, she will solve the program:

$$\begin{aligned}
r_i(v) \in \arg \min_{z \in \{0,1\}^n} & \sum_k z_k v_k \\
\text{subject to} & \sum_{k \neq i} z_k = q - 1 \text{ and } z_i = 1.
\end{aligned} \tag{7}$$

Let Γ_i denote the set of minimizers of (7). Each $\gamma_i = (\gamma_{ij})_{j=1}^n \in \Gamma_i$ corresponds to a pure proposal, since an SSP proposal in pure strategies by player i can be identified by the $(n - 1)$ -dimensional vector which specifies the players to whom player i offers their continuation payoff. A minimizer of (6), however, does not necessarily correspond to a pure proposal. Rather, it corresponds to a mixed proposal, where player i randomizes over the proposals corresponding to the elements in Γ_i (possibly with degenerate probabilities). In equilibrium, player i randomizes over the proposals corresponding to the elements in Γ_i since any proposal corresponding to an element in Γ_i yields the lowest possible payoff to player i . It is straightforward to verify that $r_{ij}(v)$ is a minimizer of (6) if and only if there exists a probability distribution $\pi_i(\cdot)$ over Γ_i such that

$$r_{ij}(v) = \sum_{\gamma_i \in \Gamma_i} \gamma_{ij} \pi_i(\gamma_i).$$

In other words, randomizing over pure proposals is payoff equivalent to offering mixed proposals. Intuitively, $r_{ij}(v)$ denotes the probability that player j is offered his continuation payoff when player i is the proposer who proposes an allocation that will be accepted.

If player i offers an allocation that is accepted, this allocation yields the payoff $y_i - \sum_{j \neq i} r_{ij}(v)v_j$ to the proposer and it yields the expected payoff $r_{ij}(v)v_j$ to player j . If no proposal is accepted, then all the players receive their continuation payoffs. Given our convention that $r_{ii}(v) = 1$, a payoff maximizing proposer i offers an allocation that will be accepted if $y_i - \sum_j r_{ij}(v)v_j > 0$, passes if $y_i - \sum_j r_{ij}(v)v_j < 0$, and is indifferent between proposing an allocation that will be accepted and passing if $y_i - \sum_j r_{ij}(v)v_j = 0$. Recall that $\alpha_i(v)$ denotes the probability that player i proposes an allocation that will be accepted. Then $\alpha_i(v)$ must satisfy the restrictions imposed in equation (2).

In equilibrium, the offer probabilities $r_{ij}(v)$ and proposal probabilities $\alpha_i(v)$, $i, j = 1, \dots, n$,

must induce the continuation payoffs v , that is $v = \delta E[v]$ where the expectation is taken over the proposer selection probabilities. Next, we show that this is satisfied by equation (5).

If agreement is not reached in the current period, then next period player i is the proposer with probability $\frac{1}{n}$. With probability $\alpha_i(v)$ she proposes an allocation that will be accepted in which case her payoff is $y_i - \sum_{j \neq i} r_{ji}(v)v_j$. With probability $1 - \alpha_i(v)$ player i passes and receives her continuation payoff v_i . Thus, conditional on being the proposer, next period's expected payoff for player i discounted back to the current period is

$$\delta \frac{1}{n} [\alpha_i(v)(y_i - \sum_{j \neq i} r_{ji}(v)v_j) + (1 - \alpha_i(v))v_i]. \quad (8)$$

Now consider the case when player i is not the proposer next period. With probability $\frac{1}{n}$ player $j \neq i$ is the proposer. Player j proposes an allocation that will be accepted with probability $\alpha_j(v)$ in which case the expected payoff to player i is $r_{ji}(v)v_i$. With probability $1 - \alpha_j(v)$ player j passes in which case player i receives her continuation payoff v_i . Thus, conditional on not being the proposer, next period's expected payoff for player i discounted back to the current period is

$$\delta \sum_{j \neq i} \frac{1}{n} [\alpha_j(v)r_{ji}(v) + (1 - \alpha_j(v))]v_i. \quad (9)$$

Combining (8) and (9) and rearranging, the continuation payoff for player i is given by equation (5).

To complete the proof consider the following strategy. When player i is not the proposer, she accepts any proposal if and only if the proposal gives her at least v_i . When player i is the proposer, she proposes an allocation with probability $\alpha_i(v)$ and passes with probability $1 - \alpha_i(v)$. If she proposes an allocation, the allocation she proposes is x with probability $\pi(\gamma_i)$, where $x_i = y_i - \sum_{j \neq i} \gamma_{ij}v_j$, $x_j = \gamma_{ij}v_j$ for all $j \neq i$, and $\pi_i(\cdot)$ is the probability distribution on Γ_i that induces the offer probabilities $r_{ij}(v)$. Clearly, this strategy implements the payoffs given by (5) and no player has an incentive to unilaterally deviate from it. ■

The expression in brackets on the right hand side of equation (5) is the expected payoff

to player i . With probability $1/n$ player i is the proposer. If she decides to propose an allocation that would be accepted, then she receives her surplus y_i net of the cost of her coalition partners (including herself). Otherwise, she either passes or proposes an allocation that would be rejected, and receives her continuation payoff in either case. Our convention that $r_{ii}(v) = 1$ implies that player i receives her continuation payoff—over and above the proposer’s surplus $y_i - w_i(v)$ if $\alpha_i(v) > 0$ —with probability one when she is the proposer. This happens with probability $1/n$. With probability $(n - 1)/n$, on the other hand, someone else is the proposer and player i receives her continuation payoff with probability $\sum_{j \neq i} r_{ji}(v)$. Since $\mu_i(v; r(v)) = \sum_{j=1}^n \mu_{ji}(v; r(v))$, the expression on the right hand side of equation (5) is the discounted expected payoff of player i . In equilibrium, this must equal her SSP payoff.

The next proposition establishes the existence of equilibria for any q -quota game.

Proposition 2 *There exists an SSP payoff vector for the q -quota game, for any $q \in \{1, \dots, n\}$.*

Proof. Given $\alpha = (\alpha_1, \dots, \alpha_n)$ and $r = [r_{ij}]_{i,j=1}^n$, define the mapping $A(\cdot; \alpha, r) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as $A(v; \alpha, r) = (A_1(v; \alpha, r), \dots, A_n(v; \alpha, r))$ where

$$A_i(v; \alpha, r) = \delta \left(\frac{1}{n} \alpha_i (y_i - \sum_j r_{ij} v_j) + \sum_j \frac{1}{n} [\alpha_j r_{ji} + (1 - \alpha_j)] v_i \right), \quad (10)$$

for all $i = 1, \dots, n$. Define the set-valued mapping $T(\cdot)$ on $[0, y_n]^n$ as

$$T(v) = \{g \in \mathbb{R}^n : \exists r \in r(v), \exists \alpha \in \alpha(v) \text{ such that } g = A(v; \alpha, r)\}. \quad (11)$$

By Proposition 1 and the definition of $T(\cdot)$, v is an SSP payoff vector for the q -quota game if and only if it is a fixed point of the set-valued mapping $T(\cdot)$, that is $v \in T(v)$.

Note that $T(\cdot)$ maps $[0, y_n]^n$ to non-empty subsets of $[0, y_n]^n$. It is easily seen that $T(v)$ is convex for all v since $r(\cdot)$ and $\alpha(\cdot)$ are convex valued. Furthermore, $T(\cdot)$ is upper semi-continuous since $r(\cdot)$ and $\alpha(\cdot)$ are upper semi-continuous and A is continuous in v , α and r . Finally, for all $v \in [0, y_n]^n$, $T(v)$ is a closed subset of the compact set $[0, y_n]^n$ and hence, $T(v)$ is compact. Thus, the result follows from Kakutani Fixed Point Theorem. ■

For the class of games we consider here, when $q = n$ (i.e., in the unanimity game), the results of Merlo and Wilson (1998) imply that the SSP payoff vector is unique. Uniqueness of the SSP equilibrium payoff vector is also guaranteed for any q -quota game where the surplus to be divided is the same for all players, i.e., $y_1 = \dots = y_n$ (Baron and Ferejohn (1989), Eraslan (2002)). However, in general, when $q < n$, the equilibrium payoffs need not be unique (Eraslan and Merlo (2002)).

Before presenting an example of a majority rule game with multiple equilibria, we establish some useful properties of SSP equilibria. In what follows, we let $v = (v_1, \dots, v_n)$ denote an arbitrary SSP equilibrium payoff vector for any q -quota game, $q \in \{1, \dots, n\}$.

Our first result establishes that if a delay is possible when a player is the proposer, then his or her payoff must be zero.

Lemma 1 *For all $i = 1, \dots, n$, if $\alpha_i(v) < 1$, then $v_i = 0$.*

Proof. If $\alpha_i(v) = 0$, then from (5) we obtain $v_i = \delta\mu_i(v; r(v))v_i \leq \delta v_i$. Since $v_i \geq 0$, we must have $v_i = 0$. If $0 < \alpha_i(v) < 1$, then the proposer must be indifferent between proposing and passing, and hence $y_i = w_i(v)$. Again, plugging in (5), we obtain $v_i \leq \delta v_i$, and the result follows since $v_i \geq 0$. ■

In the class of bargaining games we consider, players derive their bargaining power mainly from two sources: their ability to propose and their ability to vote against a proposal. As shown by Kalandrakis (2006), the proposal power is in general much more significant in determining a player's bargaining power. Our analysis confirms and strengthens this result. In particular, if an equilibrium admits the possibility of delay when a player is the proposer, then that player cannot make use of his or her proposal power at all, which in turn results in a complete loss of bargaining power and hence an equilibrium payoff of zero.

The above lemma implies that if $v_i > 0$, then we must have $\alpha_i(v) = 1$, which is possible when $y_i \geq w_i(v)$. The following lemma strengthens this result and shows that the inequality must be strict.

Lemma 2 *For all $i = 1, \dots, n$, $v_i > 0$ if and only if $y_i > w_i(v)$.*

Proof. First we show that if $v_i > 0$, then $y_i > w_i(v)$. Suppose not. Then, $v_i > 0$, but $y_i \leq w_i(v)$. If the inequality is strict, then $\alpha_i(v) = 0$. Thus, $\alpha_i(v)(y_i - w_i(v)) = 0$ whether or not the inequality is strict. Plugging this in (5) and rearranging we obtain $v_i = 0$ which is a contradiction.

Next we show that if $y_i > w_i(v)$, then $v_i > 0$. Since $y_i > w_i(v)$, by (2) we have $\alpha_i(v) = 1$, and thus (5) implies $v_i > 0$. ■

Equation (2) states that if the surplus available when player i is the proposer exceeds the cost of her coalition, then there is agreement when i is the proposer. Recall that we maintain the convention that each player i includes herself in her coalition. The next lemma shows that agreement is also reached if the surplus available when player i is the proposer exceeds the cost of her coalition net of the cost of including herself as a coalition partner.

Lemma 3 *For all $i = 1, \dots, n$, if $y_i > w_i(v) - v_i$, then $\alpha_i(v) = 1$.*

Proof. Suppose not. Then $y_i > w_i(v) - v_i$, but $\alpha_i(v) < 1$. By Lemma 1, $v_i = 0$, and so $y_i > w_i(v)$. But then (2) implies that $\alpha_i(v) = 1$ which is a contradiction. ■

From equation (5), which characterizes the equilibrium payoffs, it can be observed that a player's payoff depends on three endogenous factors: the probability of agreement when s/he is the proposer, the total cost of his or her coalition partners, and the probability that s/he receives his or her continuation payoff when s/he is not the proposer. Given that a player's payoff is decreasing in the total cost of his or her coalition partners, one might expect that "cheaper" players have higher total costs. The next results shows that this is not the case. To the contrary, if player i has a lower continuation payoff than player j , then the total cost of player i 's coalition partners cannot exceed the total cost of player j 's coalition partners.

Lemma 4 *For all $i, j = 1, \dots, n$, if $v_i \leq v_j$, then $w_i(v) \leq w_j(v)$.*

Proof. If $v_i = v_j$, then $w_i(v) = w_j(v)$ because otherwise one of the players would not be maximizing their payoff. If $v_i < v_j$, given the probability $r_{ji}(v)$ that j includes i in his

coalition when he is the proposer in equilibrium, let

$$\begin{aligned} \tilde{w}_j &= \min_{z \in [0,1]^{n-2}} \sum_{k \neq i,j} z_k v_k + r_{ji}(v) v_i + v_j \\ \text{subject to} \quad & \sum_{k \neq i,j} z_k = q - 1 - r_{ji}(v), \end{aligned} \tag{12}$$

and

$$\begin{aligned} \tilde{w}_i &= \min_{z \in [0,1]^{n-2}} \sum_{k \neq i,j} z_k v_k + r_{ji}(v) v_j + v_i \\ \text{subject to} \quad & \sum_{k \neq i,j} z_k = q - 1 - r_{ji}(v). \end{aligned} \tag{13}$$

Notice that $w_i(v) \leq \tilde{w}_i \leq \tilde{w}_j = w_j(v)$, where the first inequality follows from the fact that the vector $r_i(v)$ that defines $w_i(v)$ is a minimizer for a less restrictive program than (13), the second inequality follows from the fact that $r_{ji}(v) v_j + v_i \leq r_{ji}(v) v_i + v_j$, and the equality follows from the definition of $r_{ji}(v)$ and $w_j(v)$. ■

Our next result establishes that if player i is strictly less productive than player j , and if agreement occurs with positive probability when player i is the proposer, then agreement always occurs when player j is the proposer.

Lemma 5 *For all $i, j = 1, \dots, n$, if $y_i < y_j$ and $\alpha_i(v) > 0$, then $\alpha_j(v) = 1$.*

Proof. Suppose to the contrary that $y_i < y_j$ and $\alpha_i(v) > 0$, but $\alpha_j(v) < 1$. Since $\alpha_i(v) > 0$, it must be the case that $y_i \geq w_i(v)$. By Lemma 1, $\alpha_j(v) < 1$ implies $v_j = 0$. By Lemma 4, $w_j(v) \leq w_i(v)$. Since $y_j > y_i$, it follows that $y_j > w_j(v)$. But then (2) implies that $\alpha_j(v) = 1$ which is a contradiction. ■

Note that if agreement is ever reached when player i is the proposer, it must be that the surplus available net of the payments to her coalition partners must not be smaller than her own SSP payoff. Since the same coalition partners are also potentially available to any other player j when he is the proposer, it follows that if player j is more productive than player i , then agreement must always be reached when j is the proposer.

Lemma 6 For all $i = 1, \dots, n$, if $y_i > w_i(v)$, then $\alpha_j(v) = 1$ for all $j \geq i, \dots, n$.

Proof. Suppose to the contrary that $y_i > w_i(v)$, but $\alpha_j(v) < 1$ for some $j > i$. Since $y_i > w_i(v)$, then by (2) we have $\alpha_i(v) = 1$ and by (5) we have $v_i > 0$. Since $\alpha_j(v) < 1$, Lemma 1 implies $v_j = 0$. Consequently, by Lemma 4, we have $w_j(v) \leq w_i(v)$. Thus, $y_j \geq y_i > w_i(v) \geq w_j(v)$. But then (2) implies that $\alpha_j(v) = 1$ which is a contradiction. ■

The next result establishes that if an equilibrium admits the possibility of delay when player i is the proposer, which implies that her payoff is equal to zero, then the equilibrium payoff of all less productive players $j < i$ is also zero.

Lemma 7 For all $i = 1, \dots, n$, if $\alpha_i(v) < 1$, then $v_j = 0$ for all $j = 1, \dots, i$.

Proof. Suppose that $\alpha_i(v) < 1$. By Lemma 1 we must have $v_i = 0$. Now suppose to the contrary of the claim that there exists a $j < i$ such that $v_j > 0$. By Lemma 2, it must be the case that $y_j > w_j(v)$. But then, by Lemma 6, it must be the case that $\alpha_i(v) = 1$, which is a contradiction. ■

When player i is “cheaper” than player j , player j cannot be included in other players’ coalitions more often than player i . In addition, if agreement is reached with certainty when either of these players is the proposer, then it is also the case that the probability that player j receives his continuation payoff when he is not the proposer cannot be higher than the probability that player i receives her continuation payoff when she is not the proposer.

Lemma 8 For all $i, j = 1, \dots, n$, if $v_i < v_j$ and $\alpha_i(v) = \alpha_j(v) = 1$, then $\mu_i(v; r(v)) \geq \mu_j(v; r(v))$.

Proof. Since $v_i < v_j$, any player $k \neq i, j$ includes player i in his coalition at least as often as he includes player j , and so $r_{ki}(v) \geq r_{kj}(v)$. Furthermore, $r_{ji}(v) \geq r_{ij}(v)$. If not, then either player i or player j are not maximizing their payoff. Then, $\mu_{ji}(v; r(v)) = \alpha_j(v)r_{ji}(v) + (1 - \alpha_j(v)) \geq \alpha_i(v)r_{ij}(v) + (1 - \alpha_i(v)) = \mu_{ij}(v; r(v))$ and the result follows from equation (4). ■

Since the only source of asymmetry among players is their productivity, if player i is less productive than player j , then one would expect j to fare no worse than i in equilibrium. The following lemma shows that this is indeed the case.

Lemma 9 *SSP payoffs are monotone: that is, $v_i \leq v_j$ for all $i < j$, $i, j = 1, \dots, n$.*

Proof. If $y_i \leq w_i(v)$, then $v_i = 0$ and the proof is immediate. So suppose that $y_i > w_i(v)$ and suppose to the contrary that $v_j < v_i$. Since $y_i > w_i(v)$, we have by (2) that $\alpha_i(v) = 1$, and, by Lemma 6, we also have $\alpha_j(v) = 1$. Then by Lemma 8, we have $\mu_j(v; r(v)) \geq \mu_i(v; r(v))$. In addition, by Lemma 4, we have $w_i(v) \geq w_j(v)$.

Since $\alpha_i(v) = \alpha_j(v) = 1$, from (5) we have

$$v_i = \delta \left[\frac{1}{n} (y_i - w_i(v)) + \mu_i(v; r(v)) v_i \right], \quad (14)$$

$$v_j = \delta \left[\frac{1}{n} (y_j - w_j(v)) + \mu_j(v; r(v)) v_j \right]. \quad (15)$$

Subtracting (14) from (15), we obtain

$$v_j - v_i = \delta \left[\frac{1}{n} (y_j - y_i) + \frac{1}{n} (w_i(v) - w_j(v)) + \mu_j(v; r(v)) v_j - \mu_i(v; r(v)) v_i \right].$$

Since $\mu_j(v; r(v)) \geq \mu_i(v; r(v))$, we have

$$v_j - v_i \geq \delta \left[\frac{1}{n} (y_j - y_i) + \frac{1}{n} (w_i(v) - w_j(v)) + \mu_i(v; r(v)) (v_j - v_i) \right].$$

Rearranging, we have that

$$v_j - v_i \geq \delta \frac{\frac{1}{n} (y_j - y_i) + \frac{1}{n} (w_i(v) - w_j(v))}{1 - \delta \mu_i(v; r(v))} \geq 0.$$

This contradicts the assumption that $v_j < v_i$. ■

The result in Lemma 9 is obviously true if there is the possibility of no agreement when player i is the proposer, since in this case $v_i = 0$. The argument is more involved if agreement occurs with probability 1 when i is the proposer, in which case, by Lemma 6, there is also always an agreement when $j > i$ is the proposer. In this case, the payoff of either player is

determined by three factors: the surplus available when they are the proposer (which favors player j), the cost of coalition partners when they are the proposer (which favors player i if indeed $v_i \leq v_j$, and favors player j otherwise), and the probability of being included in the coalitions of other players conditional on agreement (which favors player i if indeed $v_i \leq v_j$, and favors player j otherwise). As shown in the proof above, the first factor always dominates in equilibrium.

From Lemma 1, if there is a possibility of no agreement when player q is the proposer, then $v_q = 0$ for any q -quota game. By Lemma 9, it follows that $v_i = 0$ for all $i = 1, \dots, q$. But then the cost of a winning coalition for player q is zero as he needs $q - 1$ votes in addition to his vote in order to secure acceptance of his proposal. This argument implies the following result:

Lemma 10 *In any q -quota game, $q \in \{1, \dots, n\}$, agreement is always reached when player q is the proposer and player q always receives a positive payoff: that is, $\alpha_q(v) = 1$ and $v_q > 0$.*

Proof. If $y_{q-1} > w_{q-1}(v)$, then the result follows from Lemma 6. If $y_{q-1} < w_{q-1}(v)$, there is no agreement when $q - 1$ is the proposer and $v_{q-1} = 0$ by Lemma 1. If $y_{q-1} = w_{q-1}(v)$, then by equation (5), it is also the case that $v_{q-1} = 0$. Hence, $v_i = 0$ for all $i \leq q - 1$ by Lemma 9. This implies that $w_q(v) - v_q = 0 < y_q$, and hence $\alpha_q(v) = 1$ by Lemma 3. Then, $v_q > 0$ by equation (5). ■

Lemma 6 and Lemma 10 together imply that if $k \geq q$ is the proposer in a q -quota game, $q \in \{1, \dots, n\}$, then agreement is reached. The converse is not true, however. That is, it is possible to have agreement in a q -quota game when player $q - 1$ is the proposer. In fact, as we show in the example below, there can be multiple equilibria, with different probabilities of agreement, when player $q - 1$ is the proposer.

Consider the following example. There are three players with a common discount factor δ equal to 0.9. The surplus available for distribution is $y_1 = 0.4$ when player 1 is the proposer, and $y_2 = y_3 = 1$ when either player 2 or player 3 are the proposer. The agreement rule is majority rule (i.e., $q = 2$).

By Lemma 10, agreement always occurs when player 2 is the proposer, and $v_2 > 0$. Then, by Lemma 2, $y_2 > w_2(v)$, and so, by Lemma 6, agreement always occurs when player 3 is

the proposer as well. Let p denote the probability that agreement occurs when player 1 is the proposer. Given p , let $v_i(p)$ denote the payoff to player i and $v(p) = (v_1(p), v_2(p), v_3(p))$. In equilibrium, $\alpha_1(v(p)) = p$. Lemma 9 implies that $v_2(p) = v_3(p)$ for all p .

First, consider the possibility that $p = 0$. By Lemma 1, $v_1(0) = 0$. Since $p = 0$, player 2 and player 3 receive their continuation payoffs with probability one when player 1 is the proposer, i.e., $\mu_{12}(v(0)) = \mu_{13}(v(0)) = 1$, and the cost of including player 1 in the winning coalition when either player 2 or player 3 are the proposer is zero.⁶ Hence, $\mu_{23}(v(0)) = \mu_{32}(v(0)) = 0$. Substituting in equation (5), we have:

$$\begin{aligned} v_2(0) &= 0.9 \left(\frac{1}{3} + \frac{1}{3}v_2(0) \right), \\ v_3(0) &= 0.9 \left(\frac{1}{3} + \frac{1}{3}v_3(0) \right). \end{aligned}$$

Since $v_2(0) = v_3(0) = 0.429 > y_1 = 0.4$, it is indeed optimal for player 1 to pass when he is the proposer, which is consistent with $p = 0$. Therefore, there is an equilibrium with $v = (0, 0.429, 0.429)$, $\alpha(v) = (0, 1, 1)$, $r_2(v) = (1, 1, 0)$ and $r_3(v) = (1, 0, 1)$.

Next, consider the possibility that $p = 1$, with $v_1(1) < v_2(1) = v_3(1)$. Since player 1 has the lowest SSP payoff, he is included with probability one in the winning coalition when either player 2 or player 3 are the proposer, i.e., $r_{21}(v(1)) = r_{31}(v(1)) = 1$, and the cost of including player 1 in the winning coalition is $v_1(1)$. In this case, however, when player 1 is the proposer, players 2 and 3 are no longer guaranteed to receive their SSP payoffs. Since agreement is reached with probability one, and agreement requires the consent of only one additional player in addition to the proposer, we have that $\mu_{12}(v(1)) = r_{12}(v(1))$ and $\mu_{13}(v(1)) = r_{13}(v(1))$. Substituting in equation (5), we have:

$$v_1(1) = 0.9 \left(\frac{0.4}{3} - \frac{1}{3}v_j(1) + \frac{2}{3}v_1(1) \right),$$

and

$$v_j(1) = 0.9 \left(\frac{1}{3} - \frac{1}{3}v_1(1) + \frac{1}{3}r_{1j}v_j(1) \right),$$

for $j = 2, 3$. Since $r_{12}(v(1)) + r_{13}(v(1)) = 1$, we obtain that $v_1(1) = 0.048$, $v_2(1) = v_3(1) =$

⁶To simplify notation, in the example, we omit r as an argument of μ .

$0.336 < 0.4 = y_1$, and $r_{12}(v(1)) = r_{13}(v(1)) = \frac{1}{2}$. This, in turn, implies that it is optimal for player 1 to make a proposal when he is the proposer, which is consistent with $p = 1$. Therefore, there is an equilibrium with $v = (0.048, 0.336, 0.336)$, $\alpha(v) = (1, 1, 1)$, $r_1(v) = (1, \frac{1}{2}, \frac{1}{2})$, $r_2(v) = (1, 1, 0)$ and $r_3(v) = (1, 0, 1)$.

In addition, there is a third equilibrium in which player 1 is indifferent between proposing and passing when he is the proposer. This implies that $v_2(p) = v_3(p) = 0.4$, $v_1(p) = 0$, and the cost of including player 1 in the winning coalition when either player 2 or player 3 are the proposer is zero. Notice that by symmetry $r_{12}(v(p)) = r_{13}(v(p)) = \frac{1}{2}$, otherwise Lemma 9 would be violated, possibly after re-enumerating the players. Hence, for $j = 2, 3$, $\mu_{1j}(v(p)) = pr_{1j}(v(p)) + (1 - p) = 1 - \frac{p}{2}$. Substituting in equation (5), we have:

$$v_j(p) = 0.9 \left(\frac{1}{3} + \frac{1}{3} \left(1 - \frac{p}{2} \right) v_j(p) \right)$$

for $j = 2, 3$, which in turn implies that $p = 1/3$.

Note that $v_1(p)$ is increasing in $r_{21}(v(p))$ and $r_{31}(v(p))$, but $r_{21}(v(p))$ and $r_{31}(v(p))$ are decreasing in $v_1(p)$. This implies that there is no equilibrium in which $p > 0$ and $v_1(p) = v_2(p) = v_3(p)$. Thus, the three equilibria characterized above are the only equilibria.

The intuition for this multiplicity result is as follows. When player 1 is the proposer, he needs the vote of only one other player, whereas if he passes, both players 2 and 3 receive their SSP payoff. This means the payoffs of players 2 and 3 are increasing in the probability that player 1 passes when he is the proposer. At the same time, as the payoffs of players 2 and 3 increase, it is more likely that player 1 cannot afford to have his proposal accepted, decreasing the probability that his proposal passes. This reinforcement effect makes the payoffs of players 2 and 3 even higher.

When the distribution of potential surplus across players is sufficiently dispersed, however, the SSP payoff vector of any q -quota game, $q \in \{1, \dots, n\}$, is unique as the discount factor goes to one (i.e., as the players become perfectly patient or as the time between offers and counteroffers goes to zero).

Proposition 3 *If $y_i < y_{i+1}$ for all $i = 1, \dots, q - 1$ and $y_{q-1} < \frac{y_q}{n-q+2}$, then as $\delta \rightarrow 1$, there is a unique equilibrium of the q -quota game, $q \in \{1, \dots, n\}$, in which $v_1 = \dots = v_{q-1} = 0$,*

and $v_i = \frac{y_i}{n-(q-1)}$ for all $i = q, \dots, n$.

Proof. First we show that as $\delta \rightarrow 1$, there is an equilibrium with $v_1 = \dots = v_{q-1} = 0$, and $v_i = \frac{y_i}{n-(q-1)}$ for all $i \geq q$. To verify that this is an equilibrium, note that $y_{q-1} < \frac{y_q}{n-q+2}$ implies that $y_{q-1} < w_{q-1}(v)$ and hence $\alpha_{q-1}(v) = 0$ by equation (2). Therefore, by Lemma 7, $v_1 = \dots = v_{q-1} = 0$. This, in turn, implies that $\mu_i(v; r(v)) = \frac{q}{n}$ and $w_i(v) = v_i$ for all $i \geq q$. Plugging in (5) and letting $\delta \rightarrow 1$ we obtain $v_i = \frac{y_i}{n-(q-1)}$ for all $i = q, \dots, n$.

To see that this is the only equilibrium, let $\kappa_q = \min\{i = 1, \dots, n : v_i > 0\}$ denote the player with the lowest index who receives a strictly positive payoff. Note that by Lemma 10, $v_q > 0$, so we must have $\kappa_q \leq q$. Let $m_q = \min\{i = 1, \dots, n : v_i = v_q\}$ denote the player with the lowest index whose payoff is equal to v_q , and let $n_q = \max\{i = 1, \dots, n : v_i = v_q\}$ denote the player with the highest index whose payoff is equal to v_q . Note that $m_q \leq q \leq n_q$.

We first show that $\kappa_q = m_q$ as $\delta \rightarrow 1$. Suppose not, i.e., suppose $\kappa_q < m_q$. We can then partition the set of players into four groups based on their relative cost as coalition partners. The first group, $Z = \{1, \dots, \kappa_q - 1\}$, contains all players $i = 1, \dots, \kappa_q - 1$ with $v_i = 0$, who are the cheapest coalition partners. Hence, they are always included in all coalitions by all players whose proposals are accepted with probability one. The next group, $L = \{\kappa_q, \dots, m_q - 1\}$, contains all players $i = \kappa_q, \dots, m_q - 1$ with $v_i < v_q$, who are the second cheapest group. Since $m_q - 1 < q - 1$, the players in L are also always included in all coalitions by all players whose proposals are accepted with probability one. Note, however, that acceptance of any proposal would also require including as a coalition partner at least one additional player from the set $M = \{m_q, \dots, n_q\}$, which contains all the players whose payoff is equal to v_q by definition of m_q and n_q . The last group, $H = \{n_q + 1, \dots, n\}$, contains the remaining players (if any), who are never included in any other player's coalition (except their own), since they are the most expensive coalition partners and their vote is never needed by others to reach the quorum q since $n_q + 1 > q$.

Let $V_g = \sum_{i \in g} v_i$ denote the aggregate payoff of all the players in group g , and $Y_g =$

$\sum_{i \in g} y_i$, $g = Z, L, M, H$. Since $\alpha_i(v) = 1$ for all $i \in L$, by equation (5), we have

$$\begin{aligned}
V_L &= \sum_{i \in L} \delta \left[\frac{1}{n} (y_i - w_i(v)) + \mu_i(v; r(v)) v_i \right] \\
&= \sum_{i \in L} \delta \left[\frac{1}{n} (y_i - V_L - (q - (m_q - 1))v_q) + v_i \right] \\
&= \delta \left[\frac{1}{n} Y_L - \frac{m_q - \kappa_q}{n} V_L - \frac{m_q - \kappa_q}{n} (q - (m_q - 1))v_q + V_L \right]. \tag{16}
\end{aligned}$$

Since $w_{\kappa_q}(v) = V_L + (q - (m_q - 1))v_q$, from equation (16) we obtain that as $\delta \rightarrow 1$

$$\lim_{\delta \rightarrow 1} w_{\kappa_q}(v) = \frac{Y_L}{m_q - \kappa_q}.$$

Note that $Y_L \geq (m_q - \kappa_q)y_{\kappa_q}$, and so $y_{\kappa_q} \leq \lim_{\delta \rightarrow 1} w_{\kappa_q}(v)$. If the inequality is strict, then as $\delta \rightarrow 1$, we have $\alpha_{\kappa_q}(v) = 0$ and hence $v_{\kappa_q}(v) = 0$, contradicting the definition of κ_q . If the inequality is weak, then $v_{\kappa_q}(v) = 0$, again ontradicting the definition of κ_q . Thus, we must have $\kappa_q = m_q$, and hence $L = \emptyset$.

Next we show that $m_q = q$. Since $v_{y_{m_q-1}} = 0$, we have $y_{m_q-1} \leq w_{m_q-1}(v)$ by Lemma 2. Then, by Lemma 4, we must have $w_i(v) = y_{m_q-1}$ for all $i = 1, \dots, m_q - 2$. Since $y_{m_q-2} < y_{m_q-1}$, this implies that $\alpha_i(v) = 0$ for all $i = 1, \dots, m_q - 2$. This, in turn, implies implies that as $\delta \rightarrow 1$,

$$\sum_{i=m_q}^q v_i = (q - (m_q - 1))v_q \geq \left(\frac{1}{n} \sum_{i=m_q}^q y_i + \frac{m_q - 2}{n} (q - (m_q - 1))v_q \right).$$

This follows from equation (5) and the fact that all players $i = m_q, \dots, q$ receive their continuation payoff at least with probability $\frac{m_q-2}{n}$ (that is, when players $j = 1, \dots, m_q - 2$ are selected as proposers). Hence, $v_q \geq \frac{\sum_{i=m_q}^q y_i}{(n - (m_q - 2))(q - (m_q - 1))}$. Since $v_{m_q} = v_q$ and $\alpha_{m_q}(v) = 1$, we must have $y_{m_q} \geq (q - (m_q - 1))v_q = \frac{\sum_{i=m_q}^q y_i}{n - (m_q - 2)}$. Rearranging, we obtain $(n - (m_q - 2))y_{m_q} \geq \sum_{i=m_q}^q y_i \geq y_q + (q - m_q)y_{m_q}$, which implies that $y_{m_q} \geq \frac{y_q}{n - q + 2}$. Since $y_{q-1} \geq y_{m_q}$, this is only possible if $m_q = q$. Thus, $m_q = q$ and $v_q \geq \frac{y_q}{n - q + 2}$. This implies that $w_{q-1}(v) \geq \frac{y_q}{n - q + 2} > y_{q-1}$, and hence $\alpha_{q-1}(v) = 0$. Thus, as $\delta \rightarrow 1$, we have that $v_1 = \dots = v_{q-1} = 0$ and $v_i = \frac{y_i}{n - (q-1)}$ for all $i = q, \dots, n$. ■

4 The Equity Properties of Alternative Voting Rules

In this section, we compare the equity of equilibrium payoffs (i.e., the relative inequality of the distribution of equilibrium payoffs), under alternative voting rules. As a useful term of comparison, we let G_y denote the Gini coefficient for the distribution of potential surplus across players which summarizes the fundamental heterogeneity in the bargaining environment we consider:

$$G_y = \frac{2 \sum_{i=1}^n iy_i}{n \sum_{i=1}^n y_i} - \frac{n+1}{n}. \quad (17)$$

By Lemma 9, any equilibrium payoff vector $v = (v_1, \dots, v_n)$ is monotone, and so we may define the Gini coefficient associated with that payoff vector as

$$G(v) = \frac{2 \sum_{i=1}^n iv_i}{n \sum_{i=1}^n v_i} - \frac{n+1}{n}. \quad (18)$$

In what follows, we denote an equilibrium payoff vector in a q -quota game, $q \in \{1, \dots, n\}$, with discount factor δ by $v^q(\delta) = (v_1^q(\delta), \dots, v_n^q(\delta))$. Our first result in this section shows that unanimity rule always induces equilibrium outcomes that are at least as unequal as the fundamentals. Specifically, we show that for any $\delta \in (0, 1)$, the Gini coefficient associated with the unique SSP payoff vector $v^n(\delta)$ of the unanimity rule game with discount factor δ is at least as large as the Gini coefficient for the distribution of potential surplus across players, that is, $G(v^n(\delta)) \geq G_y$ for all $\delta \in (0, 1)$.

Let $\kappa(\delta)$ denote the player with the lowest index such that the equilibrium probability an agreement is reached when she is the proposer is positive: that is, $\kappa(\delta)$ is the smallest i such that $\alpha_i(v^n(\delta)) > 0$. Under unanimity rule, all players receive their continuation payoffs regardless of the identity of the proposer, and regardless of whether agreement is reached or not. Hence, $r_{ij}(v^n(\delta)) = 1$ for all $\delta \in (0, 1)$ and for all $i, j \in \{1, \dots, n\}$. It follows that $\mu_i(v^n(\delta); r(v^n(\delta))) = 1$ for all $\delta \in (0, 1)$ and for all $i \in \{1, \dots, n\}$, and equation (5) reduces to:

$$v_i^n(\delta) = \begin{cases} 0 & \text{if } i < \kappa(\delta), \\ \delta \left[\frac{1}{n} (y_i - \sum_{j=1}^n v_j^n(\delta)) + v_i^n(\delta) \right] & \text{if } i \geq \kappa(\delta). \end{cases} \quad (19)$$

Summing over all i and rearranging, we obtain:

$$\sum_{i=1}^n v_i^n(\delta) = \sum_{i=\kappa(\delta)}^n v_i^n(\delta) = \frac{\delta \sum_{i=\kappa(\delta)}^n y_i}{(1-\delta)n + \delta(n - \kappa(\delta) + 1)}. \quad (20)$$

Substituting back in equation (19), when $\delta < 1$, we have

$$v_i^n(\delta) = \begin{cases} 0 & \text{if } i < \kappa(\delta), \\ \frac{\delta}{n(1-\delta)} \left(y_i - \frac{\delta \sum_{i=\kappa(\delta)}^n y_i}{(1-\delta)n + \delta(n - \kappa(\delta) + 1)} \right) & \text{if } i \geq \kappa(\delta). \end{cases} \quad (21)$$

Using this characterization, we can now state the following proposition:

Proposition 4 $G(v^n(\delta)) \geq G_y$ for all $\delta \in (0, 1)$, and the inequality is strict if and only if $y_i \neq y_j$ for some $i, j = 1, \dots, n$.

Proof. Throughout the proof, let $\kappa = \kappa(\delta)$, and $v^n = v^n(\delta)$. Notice that $\kappa = 1$ if $y_i = y_j$ for all $i, j = 1, \dots, n$. We first show that

$$\frac{\sum_{i=1}^n i y_i}{\sum_{i=1}^n y_i} < \frac{\sum_{i=\kappa}^n i y_i}{\sum_{i=\kappa}^n y_i} \text{ if and only if } \kappa > 1.$$

Clearly, the inequality is not satisfied if $\kappa = 1$. Thus, to establish the claim, it is sufficient to show

$$\frac{\sum_{i=1}^{\kappa-1} i y_i + \sum_{i=\kappa}^n i y_i}{\sum_{i=1}^{\kappa-1} y_i + \sum_{i=\kappa}^n y_i} < \frac{\sum_{i=\kappa}^n i y_i}{\sum_{i=\kappa}^n y_i} \text{ if } \kappa > 1.$$

This inequality holds if and only if

$$\sum_{i=1}^{\kappa-1} i y_i \sum_{i=\kappa}^n y_i < \sum_{i=\kappa}^n i y_i \sum_{i=1}^{\kappa-1} y_i,$$

which is satisfied when $\kappa > 1$ because $\sum_{i=1}^{\kappa-1} i y_i < (\kappa - 1) \sum_{i=1}^{\kappa-1} y_i$ and $\kappa \sum_{i=\kappa}^n y_i < \sum_{i=\kappa}^n i y_i$.

Recall that $v_i^n = 0$ for all $i < \kappa$. So,

$$G(v^n) = \frac{2 \sum_{i=1}^n i v_i^n}{n \sum_{i=1}^n v_i^n} - \frac{n+1}{n} = \frac{2 \sum_{i=\kappa}^n i v_i^n}{n \sum_{i=\kappa}^n v_i^n} - \frac{n+1}{n}.$$

Hence, the proof is complete if we show that

$$\frac{\sum_{i=\kappa}^n iw_i^n}{\sum_{i=\kappa}^n v_i^n} \geq \frac{\sum_{i=\kappa}^n iy_i}{\sum_{i=\kappa}^n y_i},$$

with equality if $\kappa = 1$. Letting $Y = \sum_{i=\kappa}^n y_i$, the left hand side is equal to

$$\frac{\sum_{i=\kappa}^n i \left(y_i - \frac{\delta Y}{(1-\delta)n + \delta(n-\kappa+1)} \right)}{\sum_{i=\kappa}^n \left(y_i - \frac{\delta Y}{(1-\delta)n + \delta(n-\kappa+1)} \right)} = \frac{\sum_{i=\kappa}^n iy_i - \frac{\delta}{(1-\delta)n + \delta(n-\kappa+1)} Y \sum_{i=\kappa}^n i}{Y - \frac{\delta}{(1-\delta)n + \delta(n-\kappa+1)} Y (n - \kappa + 1)}.$$

The proof follows since $(\sum_{i=\kappa}^n iy_i)(n - \kappa + 1) \geq (\sum_{i=\kappa}^n i)(\sum_{i=\kappa}^n y_i)$, with strict inequality if $y_i \neq y_j$ for some $i, j = 1, \dots, n$. ■

Next, we show that as the players become perfectly patient, the unanimity rule induces a unique equilibrium outcome that is more unequal (or less equitable) than any equilibrium outcome under any q -quota voting rule with $q < n$, including majority rule. In order to establish this result, we first show that if $y_i < y_n$, then agreement is never reached when player i is the proposer as the players become perfectly patient. Intuitively, this is a consequence of the result in Merlo and Wilson (1998) that the unique equilibrium under unanimity rule is efficient. If players are patient enough, efficiency requires agreement only to occur when the largest surplus is available for distribution. Indeed, from equation (21), it is straightforward to see that if $y_{\kappa(\delta)} < y_n$, then $v_n^n(\delta) \rightarrow \infty$ as $\delta \rightarrow 1$, contradicting that $\alpha_n(v^n(\delta)) = 1$. This proves the following lemma:

Lemma 11 $\lim_{\delta \rightarrow 1} \alpha_i(v^n(\delta)) = 0$ for all $i = 1, \dots, n$ such that $y_i < y_n$.

Let $\bar{\kappa}$ denote the player with the lowest index such that $y_i = y_n$. By Lemma 11 and equation (20), as $\delta \rightarrow 1$, we have that $y_i = y_n$ and $v_i^n(\delta) = v_n^n(\delta) \rightarrow y_n/(n - \bar{\kappa} + 1)$ for all $i \geq \bar{\kappa}$, and $v_j^n(\delta) \rightarrow 0$ for all $j < \bar{\kappa}$. This means that under the unanimity rule, as players become perfectly patient, all players except for those with the highest potential surplus receive a payoff of 0 in equilibrium. In particular, if $y_{n-1} < y_n$, agreement is reached only when player n is the proposer and he receives the entire surplus under the agreed upon allocation.

Now consider any q -quota rule and let $v^q(\delta)$ denote an equilibrium payoff vector under the q -quota rule when the discount factor is δ . Recall that for any q -quota rule, agreement is always reached when player q is the proposer, i.e., $\alpha_q(v^q(\delta)) = 1$ for all $\delta \in (0, 1)$. If $\alpha_{q-1}(v^q(\delta)) < 1$, then $w_n(v^q(\delta)) = v_n^q(\delta)$ and $\mu_n(v^q(\delta); r(v^q(\delta))) = \frac{q}{n}$. Substituting in equation (5) and rearranging, player n receives a payoff equal to

$$v_n^q(\delta) = \frac{\delta y_n}{n - \delta(q - 1)}.$$

Notice that, by Lemma 9, all players $i > \bar{\kappa}$ receive the same payoff. This implies that $\mu_i(v^q(\delta); r(v^q(\delta))) = \mu_n(v^q(\delta); r(v^q(\delta)))$ for all $i > \bar{\kappa}$. If agreement is not reached when $q - 1$ is the proposer, then we must have $\bar{\kappa} \geq q$ and $\mu_i(v^q(\delta); r(v^q(\delta))) = \frac{q}{n}$ for all $i > \bar{\kappa}$ since players $1, \dots, q - 1$ constitute the cheapest coalition partners, so each player $i > \bar{\kappa}$ receives her continuation payoff only when she is the proposer or one of the players $1, \dots, q - 1$ are the proposers (in which case there is no agreement). If instead agreement is reached when player $q - 1$ is the proposer, then not all players receive their continuation payoff when player $q - 1$ proposes, and by Lemma 9, players $\bar{\kappa}, \dots, n$ are the more expensive coalition partners and are therefore excluded with positive probability from the winning coalition when another player is the proposer. Thus, $\mu_i(v^q(\delta); r(v^q(\delta))) \leq \frac{q}{n}$. Furthermore, in this case, the cost of coalition partners for players $i > \bar{\kappa}$ is at least $v_n^q(\delta)$ (recall that i 's coalition partners include player i herself), and $v_i^q(\delta) = v_n^q(\delta)$. Consequently,

$$v_n^q(\delta) \leq \frac{\delta y_n}{n - \delta(q - 1)}.$$

Thus, if $q \leq \bar{\kappa}$, then as $\delta \rightarrow 1$, we have $\lim_{\delta \rightarrow 1} v_i^q(\delta) \leq \frac{y_n}{n - q + 1} \leq \frac{y_n}{n - \bar{\kappa} + 1} = \lim_{\delta \rightarrow 1} v_i^n(\delta)$ for all $i \geq \bar{\kappa}$, and if $q < \bar{\kappa}$, then $\lim_{\delta \rightarrow 1} v_i^q(\delta) < \lim_{\delta \rightarrow 1} v_i^n(\delta)$ for all $i \geq \bar{\kappa}$. This means that as players become perfectly patient, all players with the highest potential surplus fare better under unanimity rule than under the q -quota rule. Conversely, since $\lim_{\delta \rightarrow 1} v_i^n(\delta) = 0$ for all $i < \bar{\kappa}$, all players except those with the highest potential surplus (weakly) prefer the q -quota rule to the unanimity rule.

More generally, the next proposition states that in the limit as players become perfectly patient the unique equilibrium outcome under unanimity rule is always more unequal than

any equilibrium outcome under any q -quota rule with $q < n$ as long as $q \leq \bar{\kappa}$.

Proposition 5 *As $\delta \rightarrow 1$, $G(v^n(\delta)) \geq G(v^q(\delta))$ for any $q \leq \bar{\kappa} \leq n$ and for any equilibrium payoff vector $v^q(\delta)$ of the q -quota game, $q \in \{1, \dots, n\}$. The inequality is strict if $q < \bar{\kappa}$.*

Proof. Since $y_{\bar{\kappa}} = \dots = y_n$, in any equilibrium of any q -quota game, players $\bar{\kappa}, \dots, n$ receive the same payoff by Lemma 9: that is, for any q and δ , we have $v_i^q(\delta) = v_n^q(\delta)$ for all $i = \bar{\kappa}, \dots, n$. Since $\lim_{\delta \rightarrow 1} v_i^n(\delta) = 0$ for all $i < \bar{\kappa}$, we have

$$\lim_{\delta \rightarrow 1} G(v^n(\delta)) = \frac{2 \sum_{i=\bar{\kappa}}^n i}{n - \bar{\kappa} + 1} - \frac{n + 1}{n} = \frac{2 \sum_{i=\bar{\kappa}}^n i v_i^q(\delta')}{\sum_{i=\bar{\kappa}}^n v_i^q(\delta')} - \frac{n + 1}{n},$$

for all $\delta' \in (0, 1)$, any q and any equilibrium payoff vector $v^q(\delta')$ for the q -quota rule.

Furthermore, as in the proof of Proposition 4, for all q and for all δ'

$$\frac{\sum_{i=1}^n i v_i^q(\delta')}{\sum_{i=1}^n v_i^q(\delta')} \leq \frac{\sum_{i=\bar{\kappa}}^n i v_i^q(\delta')}{\sum_{i=\bar{\kappa}}^n v_i^q(\delta')}$$

with strict inequality if and only if $\bar{\kappa} > 1$. Thus $\lim_{\delta \rightarrow 1} G(v^n(\delta)) \geq G(v^q(\delta'))$ for all q and for all δ' . ■

An immediate implication of Proposition 5 is that when $y_n > y_{n-1}$, the equilibrium outcome under unanimity rule is strictly more unequal than any equilibrium outcome under any q -quota rule with $q < n$, in the limit as players become perfectly patient. If, in addition, no two players have the same potential surplus (i.e., $y_1 < y_2 < \dots < y_n$), and they are sufficiently different from each other, then as q increases (i.e., as the voting rule becomes increasingly more inclusive), in the limit as players become perfectly patient, the unique equilibrium of the game becomes relatively more inequitable.

Proposition 6 *If $y_{i+1} > (n - q + 2) y_i$ for every $i = 1, \dots, n - 1$ and for every $q = 1, \dots, n$, then, as $\delta \rightarrow 1$, the unique SSP payoff of each q -quota game, $q \in \{1, \dots, n\}$, is given by*

$$v_i^q = \begin{cases} 0 & \text{for } i = 1, \dots, q - 1, \\ \frac{y_i}{n - (q - 1)} & \text{for } i = q, \dots, n, \end{cases} \quad (22)$$

and $G(v^1) < G(v^2) < \dots < G(v^n)$.

Proof. From Proposition 3, we know that as $\delta \rightarrow 1$, if $y_{q-1} < \frac{y_q}{n-q+2}$ for every $q = 1, \dots, n$, then each q -quota game has a unique equilibrium where $v_1 = \dots = v_{q-1} = 0$, and $v_i = \frac{y_i}{n-(q-1)}$ for $i \geq q$. Thus,

$$\lim_{\delta \rightarrow 1} G(v^q(\delta)) = \frac{2 \sum_{i=q}^n iy_i}{n \sum_{i=q}^n y_i} - \frac{n+1}{n} < \frac{2 \sum_{i=q+1}^n iy_i}{n \sum_{i=q+1}^n y_i} - \frac{n+1}{n} = \lim_{\delta \rightarrow 1} G(v^{q+1}(\delta)).$$

Hence, as q increases, the equilibrium payoffs become strictly more unequal. ■

Note that, in this case, $G(v^n) > G(v^{n-1}) > \dots > G(v^1) = G_y$, where v^q , $q \in \{1, \dots, n\}$, denotes the equilibrium payoff vector for the q -quota game as $\delta \rightarrow 1$.

5 Concluding Remarks

In this paper, we have studied the equity properties of different voting rules in a multilateral bargaining environment where players are heterogeneous with respect to the potential surplus they bring to the bargaining table. We have shown that unanimity rule may generate equilibrium outcomes that are more unequal (or less equitable) than equilibrium outcomes under majority rule. In fact, as players become perfectly patient, we have shown that if there is enough heterogeneity, then the more inclusive the voting rule with respect to the number of votes required to induce agreement, the less equitable the equilibrium allocations.

These results follow naturally from basic insights of bargaining theory. Unanimity rule protects the rights of every player, including the most productive one. As players become perfectly patient, no other player has a potential surplus that is large enough to satisfy the demands of the most productive player (i.e., her reservation payoff), in order to induce her to accept a proposal when she is not the proposer. Hence, under unanimity rule, agreement occurs only when the most productive player is the proposer and every other player receives an equilibrium payoff of zero, since they never make a proposal that is accepted in equilibrium. In fact, one of the fundamental insights of noncooperative bargaining theory is that there is a benefit to proposing only when agreements occur. If agreement does not occur, the payoff associated with proposing in such an instance is just the expected discounted payoff of future agreements. Similarly, if a player is not the proposer, she will never be offered

more than the expected value of her future payoffs. Since future payoffs are discounted, it follows that if she ever earns a positive payoff, her highest payoff must be when she makes an acceptable offer. If none of her proposals are ever accepted, her SSP payoff must be zero.

On the other hand, under majority rule (in fact, under any q -quota rule with $q < n$), the vote of the most productive player is no longer required to reach an agreement when she is not the proposer. Hence, under a q -quota voting rule, agreement always occurs whenever a player $i \geq q$ is selected as proposer, and the most productive player loses her advantage. This “egalitarian” force makes the payoff distribution relatively more equitable.

REFERENCES

- Acemoglu, D. and J. Robinson (2012), *Why Nations Fail: The Origins of Power, Prosperity, and Poverty*, New York: Crown Publishing.
- Baccara, M. and R. Razin (2007), "Bargaining over New Ideas: The Distribution of Rents and the Stability of Innovative Firms," *Journal of the European Economic Association*, **5**, 1095-1129.
- Banks, J. and J. Duggan (2000), "A Bargaining Model of Collective Choice," *American Political Science Review*, **94**, 73-88.
- Baron, D. and J. Ferejohn (1989), "Bargaining in Legislatures," *American Political Science Review*, **83**, 1181-1206.
- Baron, D. and E. Kalai (1993), "The Simplest Equilibrium of a Majority Rule Division Game," *Journal of Economic Theory*, **61**, 290-301.
- Buchanan, J. M. and G. Tullock (1962), *The Calculus of Consent*, Ann Arbor: University of Michigan Press.
- Compte, O. and P. Jehiel (2010), "Bargaining and Majority Rules: A Collective Search Perspective," *Journal of Political Economy*, **118**, 189-221.
- Dixit, A., Grossman, G. M. and F. Gul (2000), "The Dynamics of Political Compromise," *Journal of Political Economy*, **108**, 531-568.
- Eraslan, H. (2002), "Uniqueness of Stationary Equilibrium Payoffs in the Baron-Ferejohn Model," *Journal of Economic Theory*, **103**, 11-30.
- Eraslan, H. and A. Merlo (2002), "Majority Rule in a Stochastic Model of Bargaining," *Journal of Economic Theory*, **103**, 31-48.
- Harrington, J. (1990), "The Role of Risk Preferences in Bargaining when Acceptance of a Proposal Requires less than Unanimous Approval," *Journal of Risk and Uncertainty*, **3**, 135-154.
- Kalandrakis, T. (2006), "Proposal Rights and Political Power," *American Journal of Political Science*, **50**, 441-448.
- Merlo, A. and C. Wilson (1995), "A Stochastic Model of Sequential Bargaining with Complete Information," *Econometrica*, **63**, 371-399.

Merlo, A. and C. Wilson (1998), “Efficient Delays in a Stochastic Model of Bargaining,” *Economic Theory*, **11**, 39-55.

Nunnari, S. (2012), “Dynamic Legislative Bargaining with Veto Power,” Working Paper.

Sutton, J. (1986), “Non-Cooperative Bargaining Theory: An Introduction”, *Review of Economic Studies*, **53**, 709-724.

Yildirim, H. (2007), “Proposal Power and Majority Rule in Multilateral Bargaining with Costly Recognition,” *Journal of Economic Theory*, **136**, 167-196.

Yildirim, H. (2010), “Distribution of Surplus in Sequential Bargaining with Endogenous Recognition,” *Public Choice*, **142**, 41-57.