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RISE Working Paper 15-009

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by Yoosoon Chang, Robin C. Sickles
and Wonho Song

RICE
Department of Economics
Baker Hall, MS22
6100 Main Street, Houston, Texas 77005
https://economics.rice.edu

Bootstrapping Unit Root Tests with Covariates*

Yoosoon Chang
Department of Economics
Indiana University

Robin C. Sickles
Department of Economics
Rice University

Wonho Song†
School of Economics
Chung-Ang University

Abstract

We consider the bootstrap method for the covariates augmented Dickey-Fuller (CADF) unit root test suggested in Hansen (1995) which uses related variables to improve the power of univariate unit root tests. It is shown that there are substantial power gains from including correlated covariates. The limit distribution of the CADF test, however, depends on the nuisance parameter that represents the correlation between the equation error and the covariates. Hence, inference based directly on the CADF test is not possible. To provide a valid inferential basis for the CADF test, we propose to use the parametric bootstrap procedure to obtain critical values, and establish the asymptotic validity of the bootstrap CADF test. Simulations show that the bootstrap CADF test significantly improves the asymptotic and the finite sample size performances of the CADF test, especially when the covariates are highly correlated with the error. Indeed, the bootstrap CADF test offers drastic power gains over the conventional unit root tests. Our testing procedures are applied to the extended Nelson and Plosser data set.

This version: August 27, 2014

JEL Classification: C12, C15, C22

Key words and phrases: Unit root tests, covariates, bootstrap consistency.

*We are grateful to Joon Park, Bill Brown, and David Papell for helpful discussions and comments. Chang gratefully acknowledges the financial support by the National Science Foundation through grant number SES-0453069/0730152.

†Corresponding author. Address correspondence to Wonho Song, School of Economics, Chung-Ang University, 84 Heukseok-ro, Dongjak-gu, Seoul, South Korea, or to whsong@cau.ac.kr.

1. Introduction

Conventional univariate tests for the presence of unit roots in aggregate economic time series have important implications for the conduct of domestic macro and international economic policy. These tests unfortunately have been plagued by reliance on relatively short time series with relatively low frequencies. Size distortions and low power are well-known problems with conventional testing procedures [see, e.g., Stock (1991), and Campbell and Perron (1991), Domowitz and El-Gamal (2001)]. Current macroeconomic theory provides little in the way of guidance on how to increase the power and moderate size distortions other than by increasing the length of the time series.

The literature has not been silent on the many efforts to overcome the low power of conventional unit root tests. One such contribution was made by Hansen (1995) who noted that conventional univariate unit root tests ignore potentially useful information from related time series and that the inclusion of related stationary covariates in the regression equation may lead to a more precise estimate of the autoregressive coefficient. He proposed to use the covariates augmented Dickey-Fuller (CADF) unit root test rather than conventional univariate unit root tests. He analyzed the asymptotic local power functions for the CADF t-statistic and discovered that enormous power gains could be achieved by the inclusion of appropriate covariates. His Monte Carlo study suggested that such gains were also possible in the finite sample power performances of the CADF vis-a-vis conventional ADF test.

Hansen (1995) showed that the limit distribution of the CADF test is dependent on the nuisance parameter that characterizes the correlation between the equation error and the covariates. Therefore, it is not possible to perform valid statistical inference directly using the CADF test. To deal with this inferential difficulty, Hansen (1995) suggested using critical values based on an estimated nuisance parameter.[3] His two-step procedure can be a practical solution for the implementation of the CADF test. However, relying on the estimated value of the nuisance parameter would introduce additional source of variability.

In this paper, we apply the bootstrap method to the CADF test to deal with the nuisance parameter dependency and to provide a valid basis for inference based on the CADF test. In particular, we show the consistency of the bootstrap CADF test and establish the asymptotic validity of the critical values from the bootstrap distribution of the test. We also show that the bootstrap test is valid under the conditional heteroskedasticity in the innovations. The asymptotic properties of the CADF and bootstrap CADF tests are investigated and the finite sample performances of the CADF tests are compared with various well-known univariate unit root tests. The simulations show that the CADF test based on the two-step procedure suffers from serious size distortions, especially when the covariates are highly correlated with the error, while our bootstrap CADF test significantly improves the asymptotic and the finite sample size performances of the CADF test. Moreover, the bootstrap CADF test offers dramatic power gains over the conventional unit root tests.

[3]Table 1 in Hansen (1995) provides asymptotic critical values for the CADF t-statistic for values of the nuisance parameter in steps of 0.1 via simulations. For intermediate values of the nuisance parameter, critical values are selected by interpolation.

As illustrations, we apply our covariate tests and standard unit root tests in a reexamination of the stationarity of U.S. domestic macroeconomic aggregates in the extended Nelson and Plosser data set. We investigate whether the findings of unit roots in the Nelson and Plosser data set are reversed when the more powerful covariate tests are used. We find that our new covariate test rejects the unit root hypothesis in three series in the Nelson and Plosser data set for the period 1930-1972.

The paper is organized as follows. Section 2 introduces the unit root test with covariates and derives limit theories for the sample tests. Section 3 applies the bootstrap methodology to the sample tests considered in Section 2 and establishes the asymptotic validity of the bootstrap test. Section 4 considers asymptotic powers of the bootstrap tests against the local-to-unity models. In Section 5, we conduct simulations to investigate the asymptotic and the finite sample performances of the bootstrap CADF test. Empirical applications are presented in Section 6 and Section 7 concludes. All mathematical proofs are provided in the Appendix.

2. Unit Root Tests with Covariates

2.1 Model and Assumptions

We consider the time series (y_t) given by

$$\triangle y_t = \alpha y_{t-1} + u_t \tag{1}$$

for $t = 1, \ldots, n$, where $\triangle$ is the usual difference operator.[4] We let the regression errors (u_t) in the model (1) to be serially correlated, and also allow them to be related to other stationary covariates. To define the data generating process for the errors (u_t) more explicitly, let (w_t) be an m-dimensional stationary covariates. It is assumed that the errors (u_t) are given by a p-th order autoregressive exogenous (ARX) process specified as

$$\alpha(L)u_t = \beta(L)'w_t + \varepsilon_t \tag{2}$$

where L is the lag operator, $\alpha(z) = 1 - \sum_{k=1}^{p} \alpha_k z^k$ and $\beta(z) = \sum_{k=-r}^{q} \beta_k z^k$.

We consider the test of the unit root null hypothesis $\alpha = 0$ for (y_t) given as in (1), against the alternative of the stationarity $\alpha < 0$. The initial value y_0 of (y_t) does not affect our subsequent analysis so long as it is stochastically bounded, and therefore we set it at zero for expositional brevity.

Under the null hypothesis of unit root, $\triangle y_t = u_t$, and we have from (2) that

$$\triangle y_t = \alpha y_{t-1} + \sum_{k=1}^{p} \alpha_k \triangle y_{t-k} + \sum_{k=-r}^{q} \beta_k' w_{t-k} + \varepsilon_t \tag{3}$$

which is an autoregression of $\triangle y_t$ augmented by its lagged level y_{t-1} and the leads and lags of the m stationary covariates in (w_t).

[4]We start with the simple model without the deterministic components to effectively deliver the essence of the theory. The models with the deterministic components will be considered at the end of this section.

For the subsequent analysis, we also need to be more explicit about the data generating process for the stationary variables (w_t) that are used as covariates. We assume that (w_t) is generated by an AR(ℓ) process as

$$\Phi(L)w_{t+r+1} = \eta_t$$

where $\Phi(z) = I_m - \sum_{k=1}^{\ell} \Phi_k z^k$.

To define explicitly the correlation between the covariates (w_t) and the series to be tested (y_t), we consider together the innovations (η_t) and (ε_t) that generate, respectively, the covariates (w_t) and the regression error (u_t), which in turn generates (y_t). Define

$$\xi_t = (\varepsilon_t, \eta_t')'$$

and denote by $|\cdot|$ the Euclidean norm: for a vector $x = (x_i)$, $|x|^2 = \sum_i x_i^2$ and for a matrix $A = (a_{ij})$, $|A|^2 = \sum_{i,j} a_{ij}^2$. We now lay out assumptions needed for the development of our asymptotic theory.

Assumption 2.1 We assume
(a) Let (ξ_t) be a martingale difference sequence such that $\mathbf{E}\xi_t\xi_t' = \Sigma$ and $(1/n)\sum_{t=1}^{n} \xi_t\xi_t' \to_p \Sigma$ with $\Sigma > 0$, and $\mathbf{E}|\xi_t|^\gamma < K$ for some $\gamma \geq 4$, where K is some constant depending only upon r.
(b) $\alpha(z)$, $\det(\Phi(z)) \neq 0$ for all $|z| \leq 1$.

The reader is referred to Chang and Park (2002) for more discussions on the technical conditions introduced in Assumption 2.1. Assumption 2.1 (a) allows for conditional heteroskedasticity (e.g., ARCH and GARCH) in all equations in the system including the covariates. It also states that the regression error (ε_t) in equation (3) is serially uncorrelated with (η_{t+k}) for $k \geq 1$. The condition effectively implies that the regression error (ε_t) is orthogonal to the lagged differences of the dependent variable $(\triangle y_{t-1}, \ldots, \triangle y_{t-p})$ and the leads and lags of the stationary covariates $(w_{t+r}, \ldots, w_{t-q})$, which is necessary for the regression (3) to be consistently estimated via usual least squares estimation. (See Hansen (1995) for more details.)

Under Assumption 2.1 (a), the following invariance principle holds

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{[ns]} \xi_t \to_d B(s)$$

for $s \in [0, 1]$ as $n \to \infty$. The limit process $B = (B_\varepsilon, B_\eta')'$ is an $(1 + m)$-dimensional vector Brownian motion with covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_\varepsilon^2 & \sigma_{\varepsilon\eta} \\ \sigma_{\eta\varepsilon} & \Sigma_\eta \end{pmatrix}.$$

Let $z_t = (\triangle y_{t-1}, \ldots, \triangle y_{t-p}, w_{t+r}', \ldots, w_{t-q}')'$. We assume

Assumption 2.2 $\sigma_u^2 > 0$ and $\mathbf{E}z_t z_t' > 0$.

The condition $\sigma_u^2 > 0$ ensures that the series (y_t) is I(1) when $\alpha = 0$, which is necessary to be able to interpret testing $\alpha = 0$ as testing for a unit root in (y_t). The condition $\mathbf{E}z_t z_t' > 0$ implies that the stationary regressors in (z_t) are asymptotically linearly independent, which is required along with the condition Assumption 2.1 (a) for the consistency of the LS coefficient estimates for (z_t).

2.2 Covariates Augmented Unit Root Tests

To introduce our test statistics more effectively, we first let $\hat{\alpha}_n$ be the OLS estimator of α from the covariates augmented regression (3), $\hat{\sigma}_n^2$ the usual error variance estimator, and $s(\hat{\alpha}_n)$ the estimated standard error for $\hat{\alpha}_n$. We also let

$$\hat{\alpha}_n(1) = 1 - \sum_{k=1}^{p} \hat{\alpha}_k$$

where $\hat{\alpha}_k$'s are the OLS estimators of α_k's in the CADF regression (3).

The statistics that we will consider in the paper are given by

$$S_n = \frac{n\hat{\alpha}_n}{\hat{\alpha}_n(1)}, \qquad T_n = \frac{\hat{\alpha}_n}{s(\hat{\alpha}_n)}. \tag{4}$$

Note that S_n is a test for the unit root based on the estimated unit root regression coefficient, and T_n is the usual t-statistics for testing the unit root hypothesis from the CADF regression (3). The test T_n is considered in Hansen (1995).

The limit theories for the tests S_n and T_n are given in

Theorem 2.3 Under the null hypothesis $\alpha = 0$, we have as $n \to \infty$,

$$S_n \to_d \sigma_\varepsilon \frac{\displaystyle\int_0^1 Q(s)dP(s)}{\displaystyle\int_0^1 Q(s)^2 ds}, \qquad T_n \to_d \frac{\displaystyle\int_0^1 Q(s)dP(s)}{\left(\displaystyle\int_0^1 Q(s)^2 ds\right)^{1/2}}$$

under Assumptions 2.1 and 2.2, where

$$Q(s) = \beta(1)'\Psi(1)B_\eta(s) + B_\varepsilon(s)$$

and $P(s) = B_\varepsilon(s)/\sigma_\varepsilon$.

The asymptotic distributions for both S_n and T_n are nonstandard and depend upon the nuisance parameters that characterize the correlation between the covariates and the regression error as shown in Hansen (1995).[5]

[5] Noting that the null limit distribution of the CADF t-test depends only on the correlation coefficient ρ^2, Hansen (1995, Table 1, p.1155) provides the asymptotic critical values for the CADF t-test for values of ρ^2 from 0.1 to 1 in steps of 0.1. The estimate for ρ^2 is constructed as $\hat{\rho}^2 = \hat{\sigma}_{v\varepsilon}^2/\hat{\sigma}_v^2\hat{\sigma}_\varepsilon^2$, where $v_t = \beta(L)'w_t + \varepsilon_t$, and $\hat{\sigma}_{v\varepsilon}$,$\hat{\sigma}_v^2$ and $\hat{\sigma}_\varepsilon^2$ are consistent nonparametric estimators of the corresponding parameters.

The models with deterministic components can be analyzed similarly. When the time series (x_t) with a nonzero mean is given by

$$x_t = \mu + y_t \tag{5}$$

or with a linear time trend

$$x_t = \mu + \delta t + y_t \tag{6}$$

where (y_t) is generated as in (1), we may test for the presence of the unit root in the process (y_t) from the CADF regression (3) defined with the fitted values (y_t^μ) or (y_t^τ) obtained from the preliminary regression (5) or (6). The limit theories for the CADF tests given in Theorem 2.3 extend easily to the models with nonzero mean and deterministic trends, and are given similarly with the following demeaned and detrended Brownian motions

$$Q^\mu(s) = Q(s) - \int_0^1 Q(t)dt$$

and

$$Q^\tau(s) = Q(s) + (6s-4)\int_0^1 Q(t)dt - (12s-6)\int_0^1 tQ(t)dt$$

in the place of the Brownian motion $Q(s)$.

3. Bootstrap Unit Root Tests with Covariates

In this section, we consider the bootstrap for the covariates augmented unit root tests S_n and T_n introduced in the previous section. Throughout the paper, we use the usual notation $*$ to signify the bootstrap samples, and use $\mathbf{P}^*$ and $\mathbf{E}^*$ respectively to denote the probability and expectation conditional on a realization of the original sample.

To construct the bootstrap CADF tests, we first generate the bootstrap samples for the m-dimensional stationary covariates (w_t) and the series (y_t) to be tested. We begin by constructing the fitted residuals which will be used as the basis for generating the bootstrap samples. We first let $u_t = \triangle y_t$ and fit the regression

$$u_t = \sum_{k=1}^{p} \tilde{\alpha}_k u_{t-k} + \sum_{k=-r}^{q} \tilde{\beta}_k' w_{t-k} + \tilde{\varepsilon}_t \tag{7}$$

by the usual OLS regression. It is important to base the bootstrap sampling on regression (3) with the unit root restriction $\alpha = 0$ imposed. The samples generated by regression (3) without the unit root restriction do not behave like unit root processes, and this will render the subsequent bootstrap procedures inconsistent as shown in Basawa et al. (1991).

Next, we fit the ℓ-th order autoregression of w_t as

$$w_{t+r+1} = \tilde{\Phi}_{1,n} w_{t+r} + \cdots + \tilde{\Phi}_{\ell,n} w_{t+r-\ell+1} + \tilde{\eta}_t \tag{8}$$

by the usual OLS regression. We may prefer, especially in small samples, to use the Yule-Walker method to estimate (8) since it always yields an invertible autoregression, thereby ensuring the stationarity of the process (w_t) [see, e.g., Brockwell and Davis (1991, Sections 8.1 and 8.2)].

We then generate the $(1+m)$-dimensional bootstrap samples (ξ_t^*), $\xi_t^* = (\varepsilon_t^*, \eta_t^{*\prime})'$ by resampling from the centered fitted residual vectors $(\tilde{\xi}_t)$, $\tilde{\xi}_t = (\tilde{\varepsilon}_t, \tilde{\eta}_t')'$ where $(\tilde{\varepsilon}_t)$ and $(\tilde{\eta}_t)$ are the fitted residuals from (7) and (8). That is, obtain iid samples (ξ_t^*) from the empirical distribution of

$$\left(\tilde{\xi}_t - \frac{1}{n}\sum_{t=1}^{n}\tilde{\xi}_t\right)_{t=1}^{n}.$$

The bootstrap samples (ξ_t^*) constructed as such will satisfy $\mathbf{E}^*\xi_t^* = 0$ and $\mathbf{E}^*\xi_t^*\xi_t^{*\prime} = \tilde{\Sigma}$, where $\tilde{\Sigma} = (1/n)\sum_{t=1}^{n}\tilde{\xi}_t\tilde{\xi}_t'$.

Next, we generate the bootstrap samples for (w_t^*) recursively from (η_t^*) using the fitted autoregression given by

$$w_{t+r+1}^* = \tilde{\Phi}_{1,n}w_{t+r}^* + \cdots + \tilde{\Phi}_{\ell,n}w_{t+r+1-\ell}^* + \eta_t^*$$

with appropriately chosen ℓ-initial values of (w_t^*), where $\tilde{\Phi}_k$, $1 \leq k \leq \ell$ are the coefficient estimates from the fitted regression (8). Initialization of (w_t^*) is unimportant for our subsequent theoretical development, we may therefore simply choose zeros for the initial values.

Then we obtain $(w_{t+r}^*, \ldots, w_{t-q}^*)$ from the sequence (w_t^*), and construct the bootstrap samples (v_t^*) as

$$v_t^* = \sum_{k=-r}^{q}\tilde{\beta}_k' w_{t-k}^* + \varepsilon_t^*$$

using the LS estimates $\tilde{\beta}_k$, $-r \leq k \leq q$ from the fitted regression (7). Then we generate (u_t^*) recursively from (v_t^*) using the fitted autoregression given by

$$u_t^* = \tilde{\alpha}_1 u_{t-1}^* + \cdots + \tilde{\alpha}_p u_{t-p}^* + v_t^*$$

with appropriately chosen p-initial values of (u_t^*), and where $\tilde{\alpha}_k$, $1 \leq k \leq p$ are the estimates for α_k's from the fitted regression (7).

Finally, we generate (y_t^*) from (u_t^*) with the null restriction $\alpha = 0$ imposed. This is to ensure the nonstationarity of the generated bootstrap samples (y_t^*), which is claimed under the null hypothesis, and to make the subsequent bootstrap tests valid. Thus we obtain (y_t^*) as

$$y_t^* = y_{t-1}^* + u_t^* = y_0^* + \sum_{k=1}^{t} u_k^*$$

which also requires initialization y_0^*. An obvious choice would be to use the initial value y_0 of (y_t), and generate the bootstrap samples (y_t^*) conditional on y_0. The choice of initial value may affect the finite sample performance of the bootstrap; however, the effect of the initial value becomes negligible asymptotically as long as it is stochastically bounded. If the mean or linear time trend is maintained as in (5) or (6) and the unit root test is performed using the demeaned or detrended data, the effect of the initial value y_0^* of the bootstrap sample would disappear. We may therefore just set $y_0^* = 0$ for the subsequent development of our theory in this section.

To construct the bootstrapped tests, we consider the following bootstrap version of the covariates augmented regression (3), which was used to construct the sample CADF tests S_n and T_n in the previous section

$$\triangle y_t^* = \alpha y_{t-1}^* + \sum_{k=1}^{p} \alpha_k \triangle y_{t-k}^* + \sum_{k=-r}^{q} \beta_k' w_{t-k}^* + \varepsilon_t^*. \tag{9}$$

We test for the unit root hypothesis $\alpha = 0$ in (9) using the bootstrap versions of the CADF tests, defined in (10) below, that are constructed analogously as their sample counterparts S_n and T_n defined in (4). Similarly as before, we denote by $\hat{\alpha}_n^*$ and $s(\hat{\alpha}_n^*)$ respectively the OLS estimator for α and the estimated standard error for $\hat{\alpha}_n^*$ obtained from the CADF regression (9) based on the bootstrap samples, and by $\hat{\alpha}_n^*(1)$ the bootstrap counterpart to $\hat{\alpha}_n(1)$. The bootstrap tests use the statistics defined as

$$S_n^* = \frac{n\hat{\alpha}_n^*}{\hat{\alpha}_n^*(1)}, \qquad T_n^* = \frac{\hat{\alpha}_n^*}{s(\hat{\alpha}_n^*)} \tag{10}$$

corresponding to S_n and T_n introduced in (4) of the previous section.

To implement the bootstrap CADF tests, we repeat the bootstrap sampling for the given original sample and obtain $a_n^*(\lambda)$ and $b_n^*(\lambda)$ such that

$$\mathbf{P}^* \left\{ S_n^* \leq a_n^*(\lambda) \right\} = \mathbf{P}^* \left\{ T_n^* \leq b_n^*(\lambda) \right\} = \lambda$$

for any prescribed size level λ. The bootstrap CADF tests reject the null hypothesis of a unit root if

$$S_n \leq a_n^*(\lambda), \; T_n \leq b_n^*(\lambda).$$

It will be shown under appropriate conditions that the tests are asymptotically valid, i.e., they have asymptotic size λ.

The following Theorem 3.1 shows that the bootstrap statistics S_n^* and T_n^* have the same null limiting distributions as the corresponding sample statistics S_n and T_n. It implies, in particular, that the bootstrap CADF tests are asymptotically valid.

Theorem 3.1 Under the null hypothesis $\alpha = 0$, we have as $n \to \infty$,

$$S_n^* \to_{d^*} \sigma_\varepsilon \frac{\int_0^1 Q(s)dP(s)}{\int_0^1 Q(s)^2 ds} \text{ in } \mathbf{P}, \quad T_n^* \to_{d^*} \frac{\int_0^1 Q(s)dP(s)}{\left(\int_0^1 Q(s)^2 ds\right)^{1/2}} \text{ in } \mathbf{P}$$

under Assumptions 2.1 and 2.2 where $Q(s)$ and $P(s)$ are defined in Theorem 2.3, in the sense of, e.g., Remark 2 in Chang and Park (2003).

Our bootstrap theory here easily extends to the tests for a unit root in models with deterministic trends, such as those introduced in (5) or (6). It is straightforward to establish the bootstrap consistency for the CADF tests applied to the demeaned and detrended time series, using the results obtained in this section.

We may also readily bootstrap the CADF tests for models generated with ARCH and GARCH innovations. In the paper, we only consider a simple model with ARCH(1) innovations. The required bootstrap procedures for models with more general ARCH and GARCH innovations are very similar. We let

$$\varepsilon_t = \epsilon_t \sqrt{h_t} \tag{11}$$

with $h_t = \pi + \lambda \varepsilon_{t-1}^2$, where $\pi > 0$, $\lambda > 0$ are unknown parameters and (ϵ_t) is a sequence of iid random variables with $\mathbf{E}\epsilon_t = 0$ and $\mathbf{E}\epsilon_t^2 = 1$. In this case, we may estimate π and λ consistently by $\tilde{\pi}$ and $\tilde{\lambda}$ in the regression $\tilde{\varepsilon}_t^2 = \tilde{\pi} + \tilde{\lambda}\tilde{\varepsilon}_{t-1}^2$, for $t = 1, \ldots, n$, using the fitted residuals defined in (7). Subsequently, we define $\tilde{\epsilon}_t = \tilde{\varepsilon}_t / \sqrt{\tilde{h}_t}$ with $\tilde{h}_t = \tilde{\pi} + \tilde{\lambda}\tilde{\varepsilon}_{t-1}^2$ for $t = 1, \ldots, n$, and standardize $(\tilde{\epsilon}_t)$ by

$$\frac{\tilde{\epsilon}_t - \sum_{t=1}^n \tilde{\epsilon}_t / n}{\sqrt{\sum_{t=1}^n (\tilde{\epsilon}_t - \sum_{t=1}^n \tilde{\epsilon}_t / n)^2 / n}}$$

and redefine it as $(\tilde{\epsilon}_t)$. Likewise, we redefine $(\tilde{\eta}_t)$ to be the centered fitted residual from regression (8). Then we let $\tilde{\zeta}_t = (\tilde{\epsilon}_t, \tilde{\eta}_t)'$, and obtain bootstrap samples (ζ_t^*), $\zeta_t^* = (\epsilon_t^*, \eta_t^*)'$, from $(\tilde{\zeta}_t)$. To get the bootstrap samples (ξ_t^*), $\xi_t^* = (\varepsilon_t^*, \eta_t^*)'$, we only need to obtain the bootstrap samples (ε_t^*) from (ϵ_t^*), since the bootstrap samples (η_t^*) are already available. However, the bootstrap samples (ε_t^*) can be readily constructed from the bootstrap samples (ϵ_t^*) by defining $\varepsilon_t^* = \epsilon_t^* \sqrt{h_t^*}$ with $h_t^* = \tilde{\pi} + \tilde{\lambda}\varepsilon_{t-1}^{*2}$ recursively for $t = 1, \ldots, n$ conditional on $h_1^* = \tilde{\pi} + \tilde{\lambda}\tilde{\varepsilon}_0^2$.

4. Asymptotics under Local Alternatives

In this section, we consider local alternatives given by

$$H_1: \ \alpha = -\frac{c}{n} \tag{12}$$

where $c > 0$ is a fixed constant, and let (y_t) be generated by (1) and (2). The asymptotic theories for the local-to-unity models are now well established [see, e.g., Stock (1994)], and the following limit theories are easily derived from them for our model:

$$
\begin{aligned}
S_n \to_d S(c) &= -c + \sigma_\varepsilon \frac{\displaystyle\int_0^1 Q_c(s)dP(s)}{\displaystyle\int_0^1 Q_c(s)^2 ds} \\
T_n \to_d T(c) &= -\frac{c}{\sigma_\varepsilon}\left(\int_0^1 Q_c(s)^2 ds\right)^{1/2} + \frac{\displaystyle\int_0^1 Q_c(s)dP(s)}{\left(\displaystyle\int_0^1 Q_c(s)^2 ds\right)^{1/2}}
\end{aligned}
$$

where

$$Q_c(s) = Q(s) - c\int_0^1 e^{-c(s-r)} Q(r)dr$$

is Ornstein-Uhlenbeck process, which may be defined as the solution to the stochastic differential equation $dQ_c(s) = -cQ_c(s)ds + dQ(s)$, and Q is defined in Theorem 2.3.

Bootstrap theories for the local-to-unity models are established in Park (2003). Here we may follow Park (2003) to show $S_n^* \to_{d^*} S$ and $T_n^* \to_{d^*} T$ under the local alternatives (12), where S and T are the limiting null distributions of S_n and T_n given in Theorem 2.3. See Chang et al. (2013) for proofs of these results.

5. Simulations

5.1 Data Generating Process

In this section, we perform a set of simulations to investigate the performances of the bootstrap tests. For the comparison of the bootstrap tests with other well-known tests, we consider only T_n^* statistic here. For the simulations, we consider (y_t) given by the unit root model (1) with the error (u_t) generated by $u_t = \alpha_1 u_{t-1} + v_t$, where the error term (v_t) is given by

$$v_t = \beta w_t + \varepsilon_t. \tag{13}$$

We model the covariate (w_t) to follow an AR(1) process as follows:

$$w_{t+1} = \phi w_t + \eta_t. \tag{14}$$

The innovations (ξ_t), $\xi_t = (\varepsilon_t, \eta_t)'$ are randomly chosen from iid $\mathbf{N}(0, \Sigma)$, where

$$\Sigma = \begin{pmatrix} 1 & \sigma_{\varepsilon\eta} \\ \sigma_{\eta\varepsilon} & 1 \end{pmatrix}.$$

Under this setup, we have the following covariate augmented ADF regression:

$$\triangle y_t = \alpha y_{t-1} + \alpha_1 \triangle y_{t-1} + \beta w_t + \varepsilon_t. \tag{15}$$

The relative merit of constructing a unit root test from the covariate augmented regression depends on the correlation between the error $(v_t), v_t = \beta w_t + \varepsilon_t$ and the covariate (w_t). As can be seen clearly from (13) and (14), the correlation depends on two parameter values, the coefficient β on the covariate and the AR coefficient ϕ of the covariate. We thus control the degree of correlation between the error (v_t) and the covariate (w_t) through these parameters. The values of β and ϕ are allowed to vary among $\{-0.8, -0.5, 0.5, 0.8\}$. The coefficient α_1 on the lagged difference term is set at 0.2 throughout the simulations. The contemporaneous covariance $\sigma_{\varepsilon\eta}$ is set at 0.4. For the test of the unit root hypothesis, we set $\alpha = 0$ and investigate the sizes in relation to corresponding nominal test sizes. For the powers, we consider $\alpha = -0.10$.[6] To investigate the effect of the presence of conditional heteroskedasticity, we also consider ARCH(1) innovations (ε_t) generated as in (11) with $\pi = 0.6$ and $\lambda = 0.4$. In this case, we let the covariance between (ϵ_t) and (η_t) be given by $\sigma_{\epsilon\eta} = 0.415$.[7] This yields the covariance between (ε_t) and (η_t), $\sigma_{\varepsilon\eta} = 0.4$, exactly as in our earlier iid simulations. We set the initial value of (h_t) at $h_1 = \pi = 0.6$.

5.2 Asymptotic Properties

In this section, the asymptotic size properties of the CADF and the bootstrap CADF tests are compared. The regression equation for the CADF test is based on the true model and it contains one lagged difference term and the current value of the covariate. The regression equation for covariate is estimated using the AR(1) model as in (14). To investigate the effects of conditional heteroskedasticity on the bootstrap test, we let the bootstrap test based on the iid innovations as BCADFi and the one based explicitly on the exact ARCH specification of innovations as BCADFa, and compare their performances.

Given our model specifications in Section 5.1, $\rho^2 = \sigma_{v\varepsilon}^2/(\sigma_v^2\sigma_\varepsilon^2)$ is calculated as follows:

$$\sigma_v^2 = \frac{\beta^2\sigma_\eta^2}{(1-\phi)^2} + \sigma_\varepsilon^2 + \frac{2\beta\sigma_{\eta\varepsilon}}{1-\phi}$$

$$\sigma_{v\varepsilon} = \frac{\beta\sigma_{\eta\varepsilon}}{1-\phi} + \sigma_\varepsilon^2$$

where $\sigma_\varepsilon^2 = 1$, $\sigma_\eta^2 = 1$ and $\sigma_{\eta\varepsilon} = 0.4$. Then, for the parameters we consider, the true ρ^2 varies from 0 to 0.950. Now, we can compare the estimated $\hat{\rho}^2$ with true ρ^2 under the simulation setup as shown in Table 1.[8] With the iid innovations, in finite samples (for $n = 50, 100$), there are large biases in $\hat{\rho}^2$ especially when ρ^2 is low. For example, when the true ρ^2 is 0.0, the estimated $\hat{\rho}^2$ is 0.245 for $n = 50$. Since the CADF test attains potential power gains at low levels of ρ^2, the size distortions possibly caused by these biases pose serious problems to the use of the test. Moreover, these biases do not seem to vanish even for large n (e.g.,

[6] Here we use the simple terms "size" and "power" to mean "Type I error" and "rejection probability under the alternative hypothesis", respectively.

[7] To find $\sigma_{\eta\epsilon}$ yielding $\sigma_{\eta\varepsilon} = 0.4$ as in the iid simulations, we use the relationship $\sigma_{\eta\varepsilon} = \sigma_{\eta\epsilon}\mathbf{E}(\sqrt{h_t})$ and simulate $\mathbf{E}(\sqrt{h_t})$ using our specification of (ε_t) in (11).

[8] In the tables below, we provide the results only for $\phi = 0.5$ and 0.8 to save space because we have lower ρ^2 for these parameters.

$n = 500$) for some parameter values. For example, for $(\beta, \phi) = (-0.5, 0.5), (-0.8, 0.5)$, the differences between true ρ^2 and estimated $\hat{\rho}^2$ are non-negligible. This is also true for the models with the ARCH innovations. Thus, from these results, we conclude that the CADF test may suffer from size distortions that come from the imprecise estimation of ρ^2.

Next, we examine the size performances of the tests in Table 2. As the sample size increases, the overall size performances of the CADF test improves and the sizes are close to 5%. However, as mentioned above, when ρ^2 are imprecisely estimated for $(\beta, \phi) = (-0.5, 0.5), (-0.8, 0.5)$, the CADF test tends to underreject. Moreover, even when we have precise estimates $\hat{\rho}^2$ for large n, the CADF test still shows large size distortions in some cases. For example, for $(\beta, \phi) = (-0.8, 0.8)$, $\hat{\rho}^2$ is close to zero, the true value of ρ^2, but the size of the CADF test is only 1% (for $n = 500$). Again, this is a serious drawback of the CADF test because the CADF test is the most useful in terms of good power performance at low levels of ρ^2. Therefore, when ρ^2 is low, the CADF test, which is based on $\hat{\rho}^2$, shows unreliable results even in large samples.[9]

In contrast, the bootstrap CADF test does not depend on the estimated $\hat{\rho}^2$ for choosing critical values and it uses, instead, bootstrapped critical values for the test. As shown in the Table 2, both the BCADFa and BCADFi show the similar results and the sizes of the bootstrap CADF test are more stable along various parameter values than the CADF test. In particular, for the parameters that we considered above, the bootstrap CADF test shows good size properties. For example, for $(\beta, \phi) = (-0.5, 0.8)$, the size of the BCADFi test is 4.7% while that of the CADF test is only 1.7%. The bootstrap CADF test tends to slightly overreject for some parameters such as $(\beta, \phi) = (-0.5, -0.8)$, but this is not our concern because they correspond to the cases where $\hat{\rho}^2$ is very high and the CADF test is the least useful in terms of power performance. Based on these experiments, we conclude that the bootstrap CADF test shows more reliable size performances even in large samples than the CADF test.

5.3 Finite Sample Properties

The finite sample performances of the bootstrap CADF test are compared with those of the sample CADF test computed from the regression (15) as well as other well-known unit root tests. More specifically, in addition to the CADF test, we also consider another CADF test suggested by Elliott and Jansson (2003) (called the EJ test here). This test is known to have maximal power against a point alternative. Thus, these three tests are all covariate-augmented and the comparisons of their size and power performances would be meaningful. As a benchmark, we consider the ADF test based on the usual ADF regression.

Choice of lag lengths critically affects the finite sample properties of the tests. To

[9]The simulation results where ρ^2 is estimated using the true model in a parametric way are available from the authors. In this case, the estimation of ρ^2 becomes more precise and the size distortions coming from the imprecise estimates of ρ^2 disappear as n grows, as expected (for example, in the cases of (β, ϕ)=(-0.5,0.5),(-0.8,0.5)). However, even with the more precise estimates of ρ^2, the size distortions observed in the cases of (β, ϕ)=(-0.5,0.8),(-0.8,0.8), where ρ^2 are very low, still remain, and they are 1.8% and 0.7%, respectively. Therefore, even with the more precise estimates of ρ^2, the CADF test still suffers from the size distortions problem, especially for low values of ρ^2.

investigate the effects of lag length selection on the finite sample performances of the tests, we use popular lag length selection methods. For the ADF and the CADF tests, AIC was used and, for the EJ test, BIC was used as suggested by Elliott and Jansson (2003). Maximum lag length is set at four for $n = 50$ and 100 and two for $n = 25$. For the choice of the lead and lag lengths of covariate of the CADF test, maximum lengths are set at four. The lead and lag lengths of the covariates are chosen in such a way that $\hat{\rho}^2$ is minimized.

Other well-known univariate unit root tests are also considered. Ng and Perron (2001) argue that the MIC information criterion along with GLS-detrended data yields a set of tests with desirable size and power properties. In light of their argument, we calculate the following tests based on the GLS-detrended data for both the statistic and the spectral density, and select the lag lengths by the MIC, with the lower bound zero and the upper bound given by $int(12(n/100)^{1/4})$. The tests considered are the Z_α test by Phillips and Perron (1988), MZ_α test as discussed in Ng and Perron (2001), DF^{GLS} test and feasible point optimal test (P_t) by Elliott, Rothenberg and Stock (1996), and modified point optimal test (MP_t) by Ng and Perron (2001).[10] We also considered the KPSS test by Kwiatkowski et al. (1992) which tests the null hypothesis of stationarity against the alternative of a unit root. For this test, the null and the alternative hypotheses are reversed and its results need to be carefully interpreted.[11]

All regressions include a fitted intercept, and the results when including a time trend are also provided. Sample sizes of $n = 50$, and 100 are examined for 5% nominal size tests.[12] Size-adjusted powers are reported where sizes are controlled by using the finite sample critical values. The reported results are based on 3,000 simulation iterations with the bootstrap critical values computed from 3,000 bootstrap repetitions. Each replication discards the first 100 observations to eliminate start-up effects. The finite sample sizes and powers for the tests are reported in Tables 3 to 6.

Tables 3 and 4 show the size results for the tests with the iid and the ARCH innovations, respectively. The results are very similar in both cases so we focus only on the cases with iid innovations. As can be seen clearly, the sample CADF test has quite noticeable size distortions over various parameters especially for small samples. The distortions are even larger when a time trend is included. For example, sizes are higher than 10% in many cases and in some cases it reaches 22%. In contrast, the bootstrap CADF test substantially correct the biases of the CADF test particularly when $\hat{\rho}^2$ is low. This improvement of the size performance is much conspicuous when a time trend is included and for small samples. For example, when $(\beta, \phi) = (0.8, 0.5)$ with $\hat{\rho}^2 = 0.09$ in Table 3, the size of the CADF test is 12.2% while that of the bootstrap CADF test is 7.5%.

The size performances of the EJ test is the most unstable among the considered tests. Moreover, the test becomes more unstable when a time trend is included. For example,

[10] We thank Elliott and Jansson, and Ng and Perron for sharing their codes with us. We do not provide the results of the Z_α, MZ_t, MSB and P_t tests because their performances are very similar to that of the MZ_α test.

[11] As the lag truncation parameter, eight was used following the suggestion in Kwiatkowski et al. (1992).

[12] The simulation results for $n = 25$ and for the cases of $\phi = -0.5$ and -0.8 are omitted to save the space. Full results are available from the authors upon request.

the size distortions of the EJ test for small sample are huge as high as 76.7% and, even for $n = 100$, sometimes the sizes are over 75%. Hence, the size performances of the EJ test are very unstable across various parameters.

The size performances of the ADF test are as good as those of the bootstrap CADF test. This implies that ignoring covariates does not significantly affect the size properties of the ADF test. The other tests shows reasonably good size properties as sample size increases but they show quite unstable results when a time trend is included. Thus, for some parameters, they tend to overreject and, for others, they tend to severely underreject. In summary, we conclude that only the ADF test and the bootstrap CADF test show reliable and satisfactory size performances.

Tables 5 and 6 show the results of powers with the iid and the ARCH innovations, respectively. Again, the tables show very similar results in both cases. The significant improvement in the finite sample sizes that the bootstrap CADF test offers does not come at the expense of finite sample powers. Indeed, the results show that the bootstrap CADF test offers drastic power gains over the conventional ADF test when $\hat{\rho}^2$ is low, where the covariates tests are expected to improve the power properties. The powers of the bootstrap CADF are more than two or three times as large as those of the other tests when $\hat{\rho}^2$ is low. Moreover, the powers of the bootstrap CADF test are comparable to those of the CADF test and sometimes even larger than those of the CADF test especially when a time trend is included. The EJ test has the highest nominal powers but its size-adjusted powers are similar to those of the CADF tests.

In contrast, the other tests show considerably lower powers compared with those of three covariate-based tests. The powers of the ADF test are lowest among the tests considered and the other tests show similar power performances. In particular, when a time trend is included, the other tests substantially lose powers. These results show that using covariates may bring enormous power gains over other univariate unit root tests.

The KPSS test tests the null hypothesis of stationarity against the alternative of a unit root. Hence, the results in the Table 5 and 6 for the stationary series present the size performances and those in the Table 3 and 4 for the unit root series present the power properties of the KPSS test. The KPSS test severely overrejects the null hypothesis, sometimes over 40%. The powers of the KPSS test are very low around 40% and especially so when a time trend is included. Thus, under the current setup of the simulations, the size and the power properties of the KPSS test are less reliable.

Our simulation results in Tables 4 and 6 for the models with ARCH innovations show that the presence of conditional heteroskedasticity does not have any major impact on our bootstrap tests even in finite samples. Our theory implies that the usual bootstrap assuming iid innovations is asymptotically valid also for models with conditional heteroskedasticity, and therefore, it is well predicted that the CADF tests relying on the iid bootstrap work for our simulation models generated by ARCH innovations in large samples. It turns out that even in finite samples the iid bootstrap works as well as the bootstrap based explicitly on the exact ARCH specification of innovations in the model. In particular, it seems clear that the ARCH bootstrap does not provide any asymptotic refinement for our tests. This is not surprising because the asymptotic distributions of the CADF tests are not pivotal.

In summary, the bootstrap CADF test has good size and power properties for all combinations of parameter values and time series dimensions and is robust to the inclusion of a time trend. From all these observations, we conclude that the bootstrap CADF test has the best size and power properties under our simulation setup.

6. Empirical Applications

In our empirical applications, we consider the Nelson and Plosser (1982) data set extended by Schotman and Van Dijk (1991). Nelson and Plosser (1982) studied the time series properties of fourteen series and found that all of them, except the unemployment series, were characterized by stochastic nonstationarity.

The testing strategy is as follows. We use lagged differences of each series for covariates, thus, only stationary covariates will be utilized in our multivariate tests.[13] Among the candidates for covariates we choose the one which gives us the smallest $\hat{\rho}^2$ since this covariate provides the most powerful test, as shown in Section 5. The lags of the differenced dependent variable are selected using the Akaike Information Criterion (AIC) with the maximum lag length four.[14] For the CADF tests, current covariate is included and the combinations of past and future covariates are tried up to the lag length four, among which the lag lengths with the smallest $\hat{\rho}^2$ are chosen. For the bootstrap tests, we use critical values computed from 5,000 bootstrap iterations. All variables in the data set are measured annually in natural logarithms. The estimated period is 1929-1973 in consideration of the structural breaks in 1929 and 1973 coinciding with the onset of the Great Depression and oil shock [see Perron (1989)]. A time trend is included in the regressions. Table 7 presents the results.

For all cases the values of $\hat{\rho}^2$ are lower than 0.09, thus we should expect, based on our simulation results, more powerful test results with the CADF and the bootstrap CADF tests than with the other tests. With these new tests we can reject the null hypothesis of a unit root for five series (GNP Deflator, Wages, Money Stock, Velocity and S&P500) by the sample CADF test and three series (GNP Deflator, Money Stock and Velocity) by the bootstrap CADF test. The bootstrap test based explicitly on the ARCH innovations presents the same results as the one based on the iid innovations. Looking at the other tests, the EJ test rejects for eight series and the DF^{GLS} test rejects for two series. The other tests reject the null hypothesis for only one series or none of the series. The KPSS test rejects for only one series, implying that the other series are all stationary.

The results for other tests are not surprising because the simulation results for $n = 50$ with a time trend show that the powers of other tests are very low. Also, for such small $\hat{\rho}^2$ as our data set, the EJ test as well as the CADF test tend to severely overreject[15]. On the other hand, the bootstrap CADF test shows reasonable size and power performances.

[13]Stock and Watson (1999) note that current theoretical literatures in macroeconomics provide neither intuition nor guidance on which covariates are candidates for our CADF and bootstrap CADF tests other than on the basis of stationarity.

[14]The Bayesian Information Criterion (BIC) gives almost similar results.

[15]$\hat{\rho}^2$ is low when ϕ is positive according to the simulation results. When we calculate the estimates ϕ of AR(1) lags of potential covariates, they all take large positive numbers.

Thus, we may conclude that the results from the EJ test and the CADF test are less reliable and accept the results from the bootstrap CADF test that there are three stationary series in the Nelson and Plosser data set for the considered sample period.

7. Conclusion

In this paper, we consider the bootstrap procedure for the covariate augmented Dickey-Fuller (CADF) unit root test which substantially improves the power of univariate unit root tests. Hansen (1995) originally proposed the CADF test and suggested a two-step procedure to overcome the nuisance parameter dependency problem. Here, we propose bootstrapping the CADF test in order to directly deal with the nuisance parameter dependency and base inferences on the bootstrapped critical values. We also establish the bootstrap consistency of the CADF test and show that the bootstrap CADF test is asymptotically valid.

The asymptotic properties of the CADF and bootstrap CADF tests are investigated and the finite sample performances of the CADF tests are compared with various well-known univariate unit root tests through simulations. The bootstrap CADF test significantly improves the asymptotic and the finite sample size performances of the CADF test, especially when the covariates are highly correlated with the error. Indeed, the bootstrap CADF test offers drastic power gains over the conventional ADF and other univariate tests. As illustrations, we apply the tests to the fourteen macroeconomic time series in the Nelson and Plosser data set for the post-1929 samples. The results of the bootstrap CADF test show that there are three stationary series in the Nelson and Plosser data set for the considered sample period.

8. Appendix

Proof of Theorem 2.3 We first define

$$\begin{aligned}
A_n &= \sum_{t=1}^{n} y_{t-1}\varepsilon_t - \left(\sum_{t=1}^{n} y_{t-1}z_t'\right)\left(\sum_{t=1}^{n} z_t z_t'\right)^{-1}\left(\sum_{t=1}^{n} z_t\varepsilon_t\right) \\
B_n &= \sum_{t=1}^{n} y_{t-1}^2 - \left(\sum_{t=1}^{n} y_{t-1}z_t'\right)\left(\sum_{t=1}^{n} z_t z_t'\right)^{-1}\left(\sum_{t=1}^{n} z_t y_{t-1}\right) \\
C_n &= \sum_{t=1}^{n} \varepsilon_t^2 - \left(\sum_{t=1}^{n} \varepsilon_t z_t'\right)\left(\sum_{t=1}^{n} z_t z_t'\right)^{-1}\left(\sum_{t=1}^{n} z_t\varepsilon_t\right).
\end{aligned}$$

We have from Lemma 2.1 of Park and Phillips (1989) that

$$\sum_{t=1}^{n} z_t z_t' = O_p(n), \quad \sum_{t=1}^{n} z_t\varepsilon_t = O_p(n^{1/2}), \quad \text{and} \quad \sum_{t=1}^{n} y_{t-1}z_t' = O_p(n).$$

Then it follows that

$$\begin{aligned}
\left|\left(\sum_{t=1}^{n} y_{t-1}z_t'\right)\left(\sum_{t=1}^{n} z_t z_t'\right)^{-1}\left(\sum_{t=1}^{n} z_t\varepsilon_t\right)\right| &\leq \left|\sum_{t=1}^{n} y_{t-1}z_t'\right|\left|\left(\sum_{t=1}^{n} z_t z_t'\right)^{-1}\right|\left|\sum_{t=1}^{n} z_t\varepsilon_t\right| = O_p(n^{1/2}) \\
\left|\left(\sum_{t=1}^{n} y_{t-1}z_t'\right)\left(\sum_{t=1}^{n} z_t z_t'\right)^{-1}\left(\sum_{t=1}^{n} z_t y_{t-1}\right)\right| &\leq \left|\sum_{t=1}^{n} y_{t-1}z_t'\right|\left|\left(\sum_{t=1}^{n} z_t z_t'\right)^{-1}\right|\left|\sum_{t=1}^{n} z_t y_{t-1}\right| = O_p(n) \\
\left|\left(\sum_{t=1}^{n} \varepsilon_t z_t'\right)\left(\sum_{t=1}^{n} z_t z_t'\right)^{-1}\left(\sum_{t=1}^{n} z_t\varepsilon_t\right)\right| &\leq \left|\sum_{t=1}^{n} \varepsilon_t z_t'\right|\left|\left(\sum_{t=1}^{n} z_t z_t'\right)^{-1}\right|\left|\sum_{t=1}^{n} z_t\varepsilon_t\right| = o_p(n).
\end{aligned}$$

Hence,

$$\begin{aligned}
n^{-1}A_n &= n^{-1}\sum_{t=1}^{n} y_{t-1}\varepsilon_t + o_p(1) \\
n^{-2}B_n &= n^{-2}\sum_{t=1}^{n} y_{t-1}^2 + o_p(1) \\
n^{-1}C_n &= n^{-1}\sum_{t=1}^{n} \varepsilon_t^2 + o_p(1).
\end{aligned}$$

Under the null, $\alpha = 0$ and we have from (4) that

$$S_n = \frac{nB_n^{-1}A_n}{\hat{\alpha}_n(1)} = \frac{1}{\hat{\alpha}_n(1)}\left(\frac{n^{-1}\sum_{t=1}^{n} y_{t-1}\varepsilon_t}{n^{-2}\sum_{t=1}^{n} y_{t-1}^2}\right) + o_p(1) \to_d \sigma_\varepsilon \frac{\int_0^1 Q(s)dP(s)}{\int_0^1 Q(s)^2 ds}$$

as required, due to Lemma A.2 in Chang et al. (2013). Similarly, the stated limit distribution of T_n follows directly from (4) and Lemma A.2 in Chang et al. (2013) as

$$T_n = \frac{1}{\hat{\sigma}_n}\left(\frac{A_n}{B_n^{1/2}}\right) = \frac{1}{\hat{\sigma}_n}\left(\frac{n^{-1}\sum_{t=1}^{n} y_{t-1}\varepsilon_t}{\left(n^{-2}\sum_{t=1}^{n} y_{t-1}^2\right)^{1/2}}\right) + o_p(1) \to_d \frac{\int_0^1 Q(s)dP(s)}{\left(\int_0^1 Q(s)^2 ds\right)^{1/2}}$$

since $\hat{\sigma}_n^2 \to_p \sigma_\varepsilon^2$ and $\pi(1) = 1/\alpha(1)$. ■

Proof of Theorem 3.1 The stochastic orders for the bootstrap sample moments appearing in the definitions of the bootstrap test S_n^* and T_n^* are easily obtained. We have

$$\left|\left(\sum_{t=1}^{n} z_t^* z_t^{*\prime}\right)^{-1}\right| = O_p^*(n^{-1}), \quad \left|\sum_{t=1}^{n} z_t^*\varepsilon_t^*\right| = O_p^*(n^{1/2}), \quad \left|\sum_{t=1}^{n} y_{t-1}^* z_t^{*\prime}\right| = O_p^*(n) \tag{16}$$

from the results in Lemma 3 of Chang and Park (2003), and therefore, it follows from (16) that

$$\left|\left(\sum_{t=1}^n y_{t-1}^* z_t^{*\prime}\right)\left(\sum_{t=1}^n z_t^* z_t^{*\prime}\right)^{-1}\left(\sum_{t=1}^n z_t^* \varepsilon_t^*\right)\right| \leq \left|\sum_{t=1}^n y_{t-1}^* z_t^{*\prime}\right|\left|\left(\sum_{t=1}^n z_t^* z_t^{*\prime}\right)^{-1}\right|\left|\sum_{t=1}^n z_t^* \varepsilon_t^*\right| = O_p^*(n^{1/2})$$

$$\left|\left(\sum_{t=1}^n y_{t-1}^* z_t^{*\prime}\right)\left(\sum_{t=1}^n z_t^* z_t^{*\prime}\right)^{-1}\left(\sum_{t=1}^n z_t^* y_{t-1}^*\right)\right| \leq \left|\sum_{t=1}^n y_{t-1}^* z_t^{*\prime}\right|\left|\left(\sum_{t=1}^n z_t^* z_t^{*\prime}\right)^{-1}\right|\left|\sum_{t=1}^n z_t^* y_{t-1}^*\right| = O_p^*(n).$$

Consequently, we have

$$n^{-1}A_n^* = n^{-1}\sum_{t=1}^n y_{t-1}^* \varepsilon_t^* + o_p^*(1) \tag{17}$$

$$n^{-2}B_n^* = n^{-2}\sum_{t=1}^n y_{t-1}^{*2} + o_p^*(1). \tag{18}$$

Moreover, we may easily deduce from (16) that

$$\hat{\alpha}_n^*(1) = \tilde{\alpha}_n(1) + O_p^*(n^{-1/2}) = \alpha(1) + o_p^*(1), \tag{19}$$

where $\tilde{\alpha}_n(1) = 1 - \sum_{k=1}^p \tilde{\alpha}_k$. Finally, if we define $\hat{\sigma}_n^{2*}$ to be the bootstrap counterpart of $\hat{\sigma}_n^2$, then it follows from (16) that

$$\hat{\sigma}_n^{2*} = \mathbf{E}^* \varepsilon_t^{*2} + O_p^*(n^{-1}) = \frac{1}{n}\sum_{t=1}^n (\tilde{\varepsilon}_t - \bar{\varepsilon}_n)^2 + O_p^*(n^{-1}) = \sigma_\varepsilon^2 + o_p^*(1). \tag{20}$$

Now it follows from the definitions of S_n^* and T_n^*, given in (10), and the results in (17) and (18) that

$$S_n^* = \frac{nB_n^{*-1}A_n^*}{\hat{\alpha}_n^*(1)} = \frac{1}{\hat{\alpha}_n^*(1)}\left(\frac{n^{-1}\sum_{t=1}^n y_{t-1}^* \varepsilon_t^*}{n^{-2}\sum_{t=1}^n y_{t-1}^{*2}}\right) + o_p^*(1)$$

$$T_n^* = \frac{1}{\hat{\sigma}_n^*}\left(\frac{A_n^*}{B_n^{*1/2}}\right) = \frac{1}{\hat{\sigma}_n^*}\left(\frac{n^{-1}\sum_{t=1}^n y_{t-1}^* \varepsilon_t^*}{\left(n^{-2}\sum_{t=1}^n y_{t-1}^{*2}\right)^{1/2}}\right) + o_p^*(1),$$

and the stated limit theories for S_n^* and T_n^* follow immediately from Lemma A.4 in Chang et al. (2013), and the results in (19) and (20). ■

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Table 1. Estimates of ρ^2 for Various n ($\alpha = 1$)

DGP		iid innovations			ARCH innovations			
β	ϕ	50	100	500	50	100	500	true ρ^2
0.8	0.8	0.320	0.306	0.301	0.323	0.310	0.304	0.335
0.5	0.8	0.438	0.427	0.411	0.432	0.426	0.412	0.432
-0.5	0.8	0.245	0.202	0.128	0.232	0.194	0.126	0.000
-0.8	0.8	0.066	0.035	0.007	0.068	0.038	0.008	0.026
0.8	0.5	0.522	0.533	0.543	0.515	0.530	0.543	0.556
0.5	0.5	0.688	0.692	0.692	0.678	0.688	0.691	0.700
-0.5	0.5	0.617	0.635	0.625	0.594	0.619	0.620	0.300
-0.8	0.5	0.327	0.323	0.278	0.307	0.308	0.273	0.057

Note: The results for $n = 50$, and 100 are based on 3,000 simulation iterations and those for $n = 500$ are based on 1,000 simulation iterations.

Table 2. Asymptotic Sizes ($\alpha = 1$, $n = 500$)

DGP		iid innovations		ARCH innovations			
β	ϕ	CADF	BCADFi	CADF	BCADFa	BCADFi	true ρ^2
0.8	0.8	0.058	0.058	0.065	0.064	0.061	0.335
0.5	0.8	0.065	0.067	0.057	0.056	0.058	0.432
-0.5	0.8	0.017	0.047	0.019	0.049	0.047	0.000
-0.8	0.8	0.007	0.031	0.007	0.030	0.031	0.026
0.8	0.5	0.050	0.057	0.050	0.056	0.055	0.556
0.5	0.5	0.063	0.070	0.061	0.068	0.069	0.700
-0.5	0.5	0.027	0.050	0.023	0.044	0.041	0.300
-0.8	0.5	0.027	0.054	0.030	0.064	0.062	0.057

Note: The results are based on 1,000 simulation iterations with the bootstrap critical values computed from 1,000 bootstrap repetitions.

Table 3. Rejection Probabilities When $\alpha = 1$ (iid innovations)

β	ϕ	ADF	MZ_α	MP_t	DF^{GLS}	KPSS	EJ	CADF	BCADFi	$\hat{\rho}^2$
with a constant only										
$n = 50$										
0.8	0.8	0.094	0.115	0.100	0.058	0.235	0.354	0.073	0.065	0.036
0.5	0.8	0.081	0.078	0.067	0.037	0.228	0.386	0.092	0.079	0.094
-0.5	0.8	0.068	0.045	0.041	0.021	0.247	0.372	0.088	0.083	0.125
-0.8	0.8	0.072	0.068	0.054	0.027	0.263	0.378	0.045	0.060	0.026
0.8	0.5	0.100	0.100	0.085	0.070	0.242	0.325	0.086	0.069	0.101
0.5	0.5	0.087	0.067	0.058	0.049	0.237	0.287	0.122	0.093	0.205
-0.5	0.5	0.080	0.035	0.031	0.039	0.246	0.191	0.136	0.104	0.381
-0.8	0.5	0.081	0.037	0.030	0.035	0.277	0.272	0.097	0.087	0.189
$n = 100$										
0.8	0.8	0.071	0.065	0.054	0.038	0.379	0.134	0.053	0.068	0.032
0.5	0.8	0.058	0.051	0.045	0.033	0.388	0.204	0.075	0.075	0.102
-0.5	0.8	0.056	0.030	0.027	0.021	0.401	0.189	0.036	0.062	0.148
-0.8	0.8	0.057	0.042	0.034	0.026	0.394	0.179	0.025	0.054	0.019
0.8	0.5	0.069	0.074	0.059	0.054	0.435	0.158	0.072	0.071	0.123
0.5	0.5	0.073	0.061	0.054	0.049	0.412	0.136	0.088	0.087	0.268
-0.5	0.5	0.073	0.039	0.035	0.036	0.405	0.087	0.085	0.090	0.518
-0.8	0.5	0.071	0.047	0.044	0.044	0.412	0.127	0.050	0.073	0.237
with a time trend										
$n = 50$										
0.8	0.8	0.117	0.121	0.114	0.036	0.121	0.767	0.102	0.073	0.032
0.5	0.8	0.081	0.052	0.043	0.012	0.104	0.686	0.136	0.082	0.080
-0.5	0.8	0.075	0.020	0.017	0.007	0.117	0.579	0.125	0.085	0.102
-0.8	0.8	0.083	0.042	0.036	0.015	0.108	0.756	0.055	0.061	0.022
0.8	0.5	0.123	0.071	0.064	0.039	0.101	0.663	0.122	0.075	0.090
0.5	0.5	0.115	0.031	0.028	0.019	0.105	0.512	0.177	0.105	0.180
-0.5	0.5	0.108	0.003	0.003	0.024	0.111	0.237	0.221	0.129	0.334
-0.8	0.5	0.103	0.008	0.008	0.012	0.108	0.456	0.131	0.099	0.164
$n = 100$										
0.8	0.8	0.076	0.062	0.054	0.032	0.207	0.756	0.065	0.068	0.029
0.5	0.8	0.056	0.026	0.023	0.012	0.211	0.722	0.095	0.077	0.092
-0.5	0.8	0.047	0.014	0.013	0.006	0.216	0.526	0.046	0.075	0.134
-0.8	0.8	0.053	0.020	0.017	0.009	0.215	0.766	0.020	0.053	0.017
0.8	0.5	0.087	0.060	0.057	0.051	0.223	0.554	0.098	0.075	0.116
0.5	0.5	0.079	0.041	0.039	0.034	0.212	0.338	0.103	0.081	0.258
-0.5	0.5	0.088	0.012	0.012	0.018	0.235	0.110	0.102	0.101	0.508
-0.8	0.5	0.071	0.013	0.012	0.015	0.216	0.292	0.061	0.080	0.224

Table 4. Rejection Probabilities When $\alpha = 1$ (ARCH innovations)

β	ϕ	ADF	MZ_{α}	MP_t	DF^{GLS}	KPSS	EJ	CADF	BCADFa	BCADFi	$\hat{\rho}^2$
with a constant only											
$n = 50$											
0.8	0.8	0.095	0.126	0.110	0.057	0.239	0.331	0.069	0.060	0.062	0.034
0.5	0.8	0.080	0.089	0.076	0.036	0.232	0.383	0.086	0.075	0.078	0.088
-0.5	0.8	0.071	0.050	0.040	0.023	0.239	0.375	0.088	0.083	0.082	0.116
-0.8	0.8	0.074	0.076	0.060	0.032	0.263	0.366	0.040	0.055	0.054	0.027
0.8	0.5	0.098	0.096	0.080	0.067	0.232	0.324	0.089	0.074	0.075	0.096
0.5	0.5	0.086	0.068	0.061	0.049	0.237	0.300	0.126	0.100	0.103	0.191
-0.5	0.5	0.079	0.034	0.029	0.042	0.247	0.190	0.129	0.102	0.103	0.359
-0.8	0.5	0.081	0.038	0.030	0.032	0.261	0.279	0.093	0.089	0.088	0.177
$n = 100$											
0.8	0.8	0.072	0.068	0.061	0.044	0.378	0.139	0.049	0.062	0.062	0.032
0.5	0.8	0.058	0.052	0.046	0.032	0.384	0.203	0.068	0.073	0.070	0.098
-0.5	0.8	0.061	0.027	0.026	0.018	0.397	0.188	0.042	0.069	0.067	0.141
-0.8	0.8	0.057	0.045	0.037	0.026	0.397	0.168	0.025	0.059	0.059	0.020
0.8	0.5	0.063	0.068	0.063	0.054	0.444	0.160	0.074	0.075	0.076	0.117
0.5	0.5	0.071	0.055	0.048	0.046	0.416	0.134	0.085	0.082	0.081	0.257
-0.5	0.5	0.077	0.042	0.037	0.040	0.427	0.095	0.083	0.089	0.086	0.497
-0.8	0.5	0.069	0.046	0.042	0.043	0.412	0.137	0.050	0.071	0.071	0.224
with a time trend											
$n = 50$											
0.8	0.8	0.117	0.130	0.116	0.040	0.118	0.759	0.102	0.073	0.072	0.030
0.5	0.8	0.088	0.058	0.053	0.015	0.105	0.684	0.133	0.082	0.080	0.076
-0.5	0.8	0.074	0.026	0.022	0.007	0.116	0.591	0.115	0.084	0.084	0.094
-0.8	0.8	0.086	0.046	0.041	0.015	0.112	0.756	0.049	0.056	0.052	0.022
0.8	0.5	0.131	0.075	0.066	0.041	0.106	0.670	0.117	0.074	0.075	0.085
0.5	0.5	0.118	0.039	0.033	0.021	0.108	0.527	0.178	0.109	0.113	0.169
-0.5	0.5	0.102	0.005	0.004	0.022	0.105	0.255	0.208	0.132	0.135	0.315
-0.8	0.5	0.093	0.009	0.009	0.010	0.109	0.478	0.121	0.093	0.092	0.154
$n = 100$											
0.8	0.8	0.079	0.064	0.057	0.030	0.203	0.731	0.063	0.069	0.067	0.029
0.5	0.8	0.057	0.028	0.029	0.014	0.216	0.722	0.088	0.075	0.077	0.089
-0.5	0.8	0.046	0.014	0.012	0.007	0.213	0.545	0.048	0.073	0.071	0.129
-0.8	0.8	0.052	0.026	0.022	0.011	0.212	0.758	0.020	0.058	0.056	0.018
0.8	0.5	0.086	0.060	0.057	0.050	0.221	0.563	0.092	0.071	0.072	0.110
0.5	0.5	0.080	0.040	0.038	0.031	0.205	0.349	0.100	0.082	0.081	0.247
-0.5	0.5	0.078	0.012	0.014	0.023	0.214	0.118	0.103	0.093	0.094	0.487
-0.8	0.5	0.072	0.018	0.018	0.018	0.203	0.307	0.057	0.084	0.082	0.211

Table 5. Rejection Probabilities When $\alpha = 0.9$ (iid innovations)

β	ϕ	ADF	MZ_α	MP_t	DF^{GLS}	KPSS	EJ	CADF	BCADFi	$\hat{\rho}^2$
with a constant only										
					$n = 50$					
0.8	0.8	0.070	0.069	0.080	0.133	0.261	0.638	0.776	0.717	0.036
0.5	0.8	0.076	0.064	0.077	0.122	0.274	0.520	0.532	0.532	0.092
-0.5	0.8	0.068	0.072	0.074	0.132	0.232	0.411	0.435	0.506	0.124
-0.8	0.8	0.073	0.067	0.079	0.122	0.252	0.596	0.771	0.722	0.029
0.8	0.5	0.086	0.108	0.113	0.124	0.187	0.465	0.543	0.548	0.100
0.5	0.5	0.096	0.119	0.124	0.139	0.200	0.335	0.308	0.404	0.204
-0.5	0.5	0.114	0.140	0.143	0.174	0.190	0.156	0.148	0.255	0.382
-0.8	0.5	0.114	0.123	0.124	0.154	0.179	0.317	0.293	0.396	0.190
					$n = 100$					
0.8	0.8	0.129	0.234	0.255	0.282	0.371	0.996	0.996	0.995	0.039
0.5	0.8	0.130	0.209	0.223	0.252	0.346	0.956	0.965	0.960	0.106
-0.5	0.8	0.138	0.309	0.315	0.371	0.355	0.920	0.913	0.905	0.150
-0.8	0.8	0.154	0.228	0.251	0.296	0.345	0.992	0.989	0.986	0.020
0.8	0.5	0.249	0.300	0.328	0.336	0.287	0.940	0.955	0.957	0.130
0.5	0.5	0.192	0.315	0.324	0.348	0.283	0.793	0.796	0.836	0.268
-0.5	0.5	0.245	0.363	0.391	0.415	0.270	0.502	0.376	0.474	0.514
-0.8	0.5	0.225	0.293	0.301	0.324	0.278	0.765	0.760	0.759	0.236
with a time trend										
					$n = 50$					
0.8	0.8	0.058	0.036	0.036	0.084	0.246	0.362	0.458	0.481	0.032
0.5	0.8	0.065	0.039	0.044	0.080	0.257	0.231	0.299	0.326	0.075
-0.5	0.8	0.070	0.041	0.050	0.076	0.239	0.202	0.182	0.308	0.098
-0.8	0.8	0.050	0.043	0.044	0.068	0.258	0.309	0.506	0.494	0.023
0.8	0.5	0.061	0.064	0.070	0.080	0.190	0.221	0.308	0.346	0.084
0.5	0.5	0.064	0.063	0.064	0.072	0.183	0.169	0.157	0.260	0.175
-0.5	0.5	0.069	0.086	0.088	0.097	0.163	0.094	0.071	0.198	0.335
-0.8	0.5	0.079	0.080	0.083	0.104	0.180	0.182	0.171	0.294	0.161
					$n = 100$					
0.8	0.8	0.089	0.087	0.097	0.149	0.439	0.943	0.989	0.985	0.036
0.5	0.8	0.107	0.117	0.131	0.167	0.422	0.813	0.911	0.906	0.096
-0.5	0.8	0.107	0.153	0.156	0.202	0.396	0.722	0.821	0.837	0.133
-0.8	0.8	0.110	0.130	0.137	0.184	0.423	0.923	0.974	0.967	0.017
0.8	0.5	0.135	0.143	0.152	0.160	0.338	0.775	0.879	0.893	0.121
0.5	0.5	0.125	0.160	0.163	0.176	0.323	0.575	0.657	0.716	0.252
-0.5	0.5	0.146	0.184	0.180	0.225	0.311	0.284	0.235	0.380	0.498
-0.8	0.5	0.133	0.144	0.153	0.176	0.329	0.537	0.590	0.665	0.219

Table 6. Rejection Probabilities When $\alpha = 0.9$ (ARCH innovations)

β	ϕ	ADF	MZ_α	MP_t	DF^{GLS}	KPSS	EJ	CADF	BCADFa	BCADFi	$\hat{\rho}^2$
with a constant only											
					$n = 50$						
0.8	0.8	0.070	0.066	0.078	0.133	0.262	0.645	0.798	0.732	0.734	0.036
0.5	0.8	0.072	0.072	0.079	0.122	0.268	0.544	0.580	0.566	0.565	0.086
-0.5	0.8	0.066	0.079	0.095	0.145	0.235	0.413	0.427	0.514	0.516	0.118
-0.8	0.8	0.078	0.064	0.069	0.130	0.249	0.596	0.777	0.725	0.726	0.029
0.8	0.5	0.091	0.105	0.118	0.131	0.191	0.491	0.540	0.584	0.585	0.094
0.5	0.5	0.085	0.128	0.128	0.140	0.202	0.346	0.307	0.422	0.425	0.192
-0.5	0.5	0.115	0.146	0.152	0.162	0.187	0.182	0.153	0.272	0.275	0.360
-0.8	0.5	0.105	0.128	0.128	0.155	0.178	0.323	0.302	0.427	0.425	0.176
					$n = 100$						
0.8	0.8	0.142	0.231	0.239	0.274	0.370	0.997	0.998	0.998	0.997	0.039
0.5	0.8	0.143	0.219	0.231	0.272	0.349	0.963	0.972	0.964	0.964	0.102
-0.5	0.8	0.138	0.296	0.307	0.343	0.349	0.927	0.916	0.906	0.906	0.143
-0.8	0.8	0.148	0.231	0.235	0.297	0.349	0.988	0.992	0.990	0.989	0.021
0.8	0.5	0.265	0.311	0.318	0.329	0.284	0.941	0.957	0.960	0.959	0.124
0.5	0.5	0.210	0.337	0.357	0.357	0.282	0.819	0.817	0.837	0.837	0.258
-0.5	0.5	0.246	0.370	0.376	0.398	0.275	0.518	0.393	0.486	0.485	0.496
-0.8	0.5	0.226	0.307	0.317	0.330	0.281	0.767	0.768	0.780	0.776	0.221
with a time trend											
					$n = 50$						
0.8	0.8	0.065	0.037	0.037	0.079	0.249	0.347	0.486	0.505	0.508	0.032
0.5	0.8	0.067	0.040	0.043	0.075	0.252	0.247	0.301	0.351	0.351	0.070
-0.5	0.8	0.072	0.042	0.046	0.077	0.242	0.206	0.208	0.325	0.330	0.093
-0.8	0.8	0.057	0.044	0.046	0.068	0.258	0.324	0.555	0.516	0.513	0.023
0.8	0.5	0.060	0.060	0.064	0.079	0.182	0.241	0.328	0.369	0.378	0.079
0.5	0.5	0.059	0.064	0.065	0.077	0.184	0.185	0.168	0.263	0.265	0.166
-0.5	0.5	0.080	0.085	0.089	0.093	0.161	0.103	0.079	0.212	0.217	0.312
-0.8	0.5	0.080	0.075	0.079	0.109	0.184	0.176	0.173	0.307	0.308	0.149
					$n = 100$						
0.8	0.8	0.087	0.089	0.096	0.143	0.441	0.950	0.990	0.986	0.986	0.035
0.5	0.8	0.099	0.111	0.120	0.158	0.421	0.825	0.920	0.916	0.915	0.093
-0.5	0.8	0.113	0.147	0.147	0.185	0.404	0.726	0.826	0.843	0.843	0.128
-0.8	0.8	0.111	0.123	0.130	0.173	0.421	0.921	0.977	0.967	0.966	0.018
0.8	0.5	0.142	0.135	0.152	0.150	0.337	0.807	0.882	0.903	0.903	0.116
0.5	0.5	0.110	0.153	0.166	0.184	0.329	0.594	0.665	0.731	0.734	0.243
-0.5	0.5	0.155	0.187	0.185	0.217	0.312	0.319	0.256	0.393	0.393	0.479
-0.8	0.5	0.139	0.134	0.134	0.170	0.324	0.534	0.607	0.686	0.686	0.206

Table 7. Tests for a Unit Root in the Extended Nelson-Plosser Data

Series	ADF	MZ_α	MP_t	DF^{GLS}	KPSS	EJ	CADF	BCADF	$\hat{\rho}^2$	Covariate
Real GNP	-3.201	-6.631	13.742	-2.096	0.105	61.159	1.504	1.504	0.000	Real pc GNP
Nominal GNP	-2.170	-15.509	5.958	-3.190*	0.102	26.853	0.397	0.397	0.000	Money Stock
Real p. c. GNP	-3.107	-5.724	15.919	-1.861	0.111	26.041*	-1.863	-1.863	0.000	Ind. Prod.
Indu. Prod.	-2.942	-9.365	9.733	-2.522	0.116	30.366*	-0.363	-0.363	0.000	Money Stock
Employment	-2.776	-5.780	15.762	-1.899	0.107	36.418*	-1.764	-1.764	0.000	Unemployment
Unemployment	-2.733	-5.681	15.998	-1.804	0.125	15.785*	0.454	0.454	0.000	Employment
GNP Deflator	-2.529	-5.569	16.257	-2.060	0.085	41.839*	-2.782*	-2.782	0.000	Nominal GNP
Consumer Prices	-1.852	-4.584	19.495	-1.986	0.076	52.776	-1.411	-1.411	0.000	GNP Deflator
Wages	-1.908	-4.134	21.971	-2.059	0.100	29.348	-4.618*	-4.618*	0.003	S&P 500
Real Wages	-3.122	-5.898	15.389	-1.838	0.117	7.231*	-0.500	-0.500	0.019	Real GNP
Money Stock	-2.779	-17.981*	5.118*	-3.301*	0.100	8.562*	-3.795*	-3.795*	0.000	Nominal GNP
Velocity	-1.990	-8.645	10.709	-2.494	0.103	4.185*	-4.094*	-4.094*	0.002	Nominal GNP
Bond Yield	-2.375	-1.534	44.501	-1.098	0.165*	21.649	-1.537	-1.537	0.085	Nominal GNP
S&P 500	-1.657	-4.912	18.499	-2.245	0.128	15.962	-2.422*	-2.422	0.008	Money Stock
No of Rejections	0.000	1.000	1.000	2.000	1.000	8.000	5.000	3.000		

Note: A constant and a time trend are included. * indicates rejection at the 5% level. Bootstrap repetition is 5000.